H^s-CLASS CONSTRUCTION OF AN ALMOST INVARIANT SLOW SUBSPACE FOR THE KLEIN–GORDON EQUATION IN THE NON-RELATIVISTIC LIMIT

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ABSTRACT. We consider the linear Klein–Gordon equation in one spatial dimension with periodic boundary conditions in the non-relativistic limit where $\varepsilon = \hbar^2/(mc^2)$ tends to zero. It is classical that the equation is well posed, for example, in the sense of possessing a continuous semiflow into spaces $H^{s+1} \times H^s$ for wave function and momentum, respectivly. In this paper, we iteratively contruct a family of bounded operators $F^N_{\text{slow}} \colon H^{s+1} \to H^s$ whose graphs are $O(\varepsilon^N)$-invariant subspaces under the Klein–Gordon evolution for $O(1)$ times. Contrary to a naive asymptotic series, there is no "loss of derivatives" in the iterative step, i.e., the Sobolev index s can be chosen independent of N. This is achieved by solving an operator Sylvester equation at each step of the construction.

1. INTRODUCTION

We study the linear Klein–Gordon equation with periodic boundary conditions in the form

$$\varepsilon\,\partial_t^2 u - \mathrm{i}\,\partial_t u - \Delta u = Mu\,, \tag{1}$$

where $u\colon [0,T] \times \mathbb{T} \to \mathbb{C}$, $\Delta u = \partial_x^2 u$ denotes the Laplacian, and M is a linear operator, bounded on $H^s(\mathbb{T})$ and on $H^{s+1}(\mathbb{T})$ for some $s \in \mathbb{R}$. In the interesting cases, M does not commute with the Laplacian.

Equation (1) is obtained from the classical Klein–Gordon equation for the wave function ψ of a single spinless relativistic particle of mass m,

$$\frac{\hbar^2}{2mc^2}\,\partial_t^2\psi - \frac{\hbar^2}{2m}\,\Delta\psi + \left(\frac{mc^2}{2} - M\right)\psi = 0\,, \tag{2}$$

via the rotating wave ansatz

$$\psi = u\,\exp(\mathrm{i}mc^2 t/\hbar)\,, \tag{3}$$

where $\hbar$ is the Planck constant and c is the speed of light. Chossing units such that m and $\hbar$ take value one and setting $\varepsilon = 1/(2c^2)$, we note that $\varepsilon \to 0$ corresponds to the non-relativistic limit $c \to \infty$.

In the following, we shall write (1) as a first order system of evolution equations

$$\partial_t u = v\,, \tag{4a}$$

$$\varepsilon\,\partial_t v = \mathrm{i}\,v + \Delta u + Mu\,. \tag{4b}$$

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This system has a formal resemblance with the finite-dimensional two-scale fast-slow system

$$\dot{q} = p \,, \tag{5a}$$

$$\varepsilon \, \dot{p} = Jp - \nabla V(q) \,, \tag{5b}$$

where $q \colon [0,T] \to \mathbb{R}^{2d}$ is the vector of positions, p is the vector of corresponding momenta, J is a symplectic matrix, and V is a smooth potential. In this setting, there is an asymptotic separation of time scales: the linearized dynamics has d "slow" eigenvalues of magnitude $O(1)$ and d "fast" eigenvalues of magnitude $O(\varepsilon^{-1})$. It is straightforward to construct an almost invariant slow manifold to $O(\varepsilon^N)$, and it is well known how to achieve invariance up to exponentially small terms [2, 8, 10]. Similar results are possible with an infinite number of fast degrees of freedom [7].

In the direct construction of an $O(\varepsilon^N)$ slow manifold $p = F^N_{\text{slow}}(q)$, one seeks an asymptotic series

$$F^N_{\text{slow}}(q) = \sum_{k=0}^{N} f_k(q) \, \varepsilon^k \,, \tag{6}$$

where the coefficient vector fields f_k are determined by the condition that the evolution equation for the fast residual component $w = p - F^N_{\text{slow}}(q)$ is of order $O(\varepsilon^{N+1})$. This will directly lead to a shadowing result of the form

$$\sup_{t \le T} \|q(t) - Q(t)\| \le c \, \varepsilon^{N+1} \,, \tag{7}$$

where q solves (5) with prepared initial data $p_0 = F^N_{\text{slow}}(q_0)$ and Q solves the slow equation

$$\dot{Q} = F^N_{\text{slow}}(Q) \tag{8}$$

with $Q(0) = q_0$.

The situation for the Klein–Gordon equation is different, even in the linear case. Since the Laplacian is unbounded, there is no spectral gap between the families, parameterized by ε, of eigenvalues that remain bounded and those that diverge as $\varepsilon \to 0$. On the level of the asymptotic construction, we observe that an iterative construction of the slow vector field as, for example, described in [3], will involve composition with the Laplacian, i.e., will lead to a loss of two derivatives in Sobolev space per iteration. This loss of derivatives is observed in related perturbation problems, e.g. [9, 12].

In this paper, we show, focusing on the linear Klein–Gordon equation (1), that it is possible to construct an infinite-dimensional analog of the slow manifold in the example above: we construct a linear "slow" subspace, defined as the graph of a linear operator from H^{s+1} into H^s, with the property that (i) it is invariant up to terms of $O(\varepsilon^{N+1})$ in the H^s topology and (ii) it emerges as a *regular* perturbation expansion in the functional setting of the Schrödinger equation which is the formal limit for $\varepsilon \to 0$ of the Klein–Gordon equation (1).

Our result is a proof-of-concept, motivated by the question whether variational constructions of slow manifolds, as have been studied in the finite-dimensional case in [3], can be justified in the context of the linear or the semi-linear Klein–Gordon equation. The direct construction here leads to a functional integral representation of the coefficient operators defining the "slow" subspace, thus cannot be computed practically. The variational construction, on the other hand, leads to vector fields which are more directly computable, but whose remainder terms are hard to express.

The approach of [3] uses the direct construction as a stepping stone to characterize the remainder of the variational slow vector field. Thus, the present result is an indication that a similar proof might be possible in infinite dimensions as well, even though additional analytical obstacles need to be overcome.

The remainder of the paper is structured as follows. We next describe the formal construction of the slow subspace. We present order conditions which, to avoid "loss of derivatives," are formulated as a sequence of operator Sylvester equations. In Section 3, we provide the solution theory for these operator equations. In the final Section 4, we formulate the main theorem, a shadowing estimate analogous to (7).

2. Formal construction of the "slow" vector field

In this section, we detail the formal construction of the family of vector fields F_{slow}^N which defines the almost invariant slow subspace. In order to avoid loss of derivatives, we make two nontrivial provisions. First, we use the *exact* splitting of the subspaces associated with the unbounded part of the equation, so that this contribution is included in description of the slow subspace to all orders. Second, we include certain next-order contributions into the previous-order terms so that the order conditions will involve *inverses* rather than direct right-hand contributions of unbounded operators.

To begin, we note that (4) with $M = 0$ is easily block-diagonalized. Its eigenoperators must satisfy the characteristic equation

$$\varepsilon \, L_\pm^2 - \mathrm{i} \, L_\pm - \Delta = 0 \, , \tag{9}$$

so that

$$L_\pm = \mathrm{i} \, \frac{1 \pm \sqrt{1 - 4\varepsilon\Delta}}{2\varepsilon} \, . \tag{10}$$

Clearly, there is no spectral gap between the two eigenoperators. However,

$$\lim_{\varepsilon \to 0} L_-(\varepsilon) \to \mathrm{i} \, \Delta \tag{11}$$

in the strong operator topology of H^s whereas the sequence $L_+(\varepsilon)u$ diverges for any nonzero $u \in H^s$. In this sense, we shall speak of the subspace of phase space associated with L_- as the *slow subspace* and the subspace associated with L_+ as the *fast subspace*.

For the full system (4), we seek an approximate description of the slow subspace as the graph of an operator of the form $L_- + F_{\text{slow}}^N$, where

$$F_{\text{slow}}^N = \sum_{k=0}^N F_k \tag{12}$$

and $\{F_k\}$ is a sequence of bounded operators on H^s. In other words, we keep the unbounded operator contribution to the slow motion to all orders and only seek for corrections related to $M \neq 0$ in an iterative fashion. Correspondingly, the fast subspace contribution is sought in the form

$$w = v - L_- u - F_{\text{slow}}^{N+1} u \, . \tag{13}$$

Then, differentiating (13) in time and invoking the characteristic equation (9) to simplify the expression, we find that (4) can be written as

$$\partial_t u = L_- u + F^N_{\text{slow}} u + \varepsilon^{N+1} \, F_{N+1} u + w \,, \tag{14a}$$

$$\begin{aligned} \partial_t w = \Big(\frac{\mathrm{i}}{\varepsilon} - L_- - F^{N+1}_{\text{slow}} \Big) w + \frac{1}{\varepsilon} \, \big(M + \mathrm{i} \, F^{N+1}_{\text{slow}} \big) u \\ - \big(L_- F^{N+1}_{\text{slow}} + F^{N+1}_{\text{slow}} L_- + F^{N+1}_{\text{slow}} F^{N+1}_{\text{slow}} \big) u \,. \end{aligned} \tag{14b}$$

The goal is is to determine F_k order by order such that the second and third term on the right of (14b) can be eliminated up to an $O(\varepsilon^{N+1})-$remainder. The slow limit equation will then be given by

$$\partial_t U = L_- U + F^N_{\text{slow}} U \,, \tag{15}$$

formally valid up to terms of $O(\varepsilon^{N+1})$. Provided we can show that the remainder is an H^s-bounded operator, the difference between solutions of (15) and the full system can be controlled via simple energy estimates.

We note that the obvious procedure—set $F_0 = \mathrm{i} \, M$ and choose each next order F_k to eliminate the contribution from the final line of (14b) coming in at this order—will still lead to unbounded operators F_k because L_- itself is only bounded as an operator from H^{s+1} into H^s. Thus, we still lose one derivative per interation of the scheme via the first two terms in the last pair of parentheses of (14b). However, this loss of derivatives could be avoided if it is possible to account for these two terms as part of the condition which determines the slow vector field at the *previous* order. Noting that

$$\mathrm{i} - \varepsilon \, L_- = \varepsilon \, L_+ \,, \tag{16}$$

we see that we must require, at $O(\varepsilon^0)$, that

$$M + \varepsilon \, L_+ F_0 - \varepsilon \, F_0 L_- = 0 \,. \tag{17a}$$

Likewise, at order $O(\varepsilon^k)$ for $k \geq 0$, we must require that

$$\varepsilon \, L_+ F_k - \varepsilon \, F_k L_- - \operatorname{coef}(F^{N+1}_{\text{slow}} F^{N+1}_{\text{slow}}, \varepsilon^k) = 0 \,. \tag{17b}$$

These modified order conditions require solving an operator Sylvester equation at each step. In the following Section 3, we shall show that the solution to (17) is given by

$$F_k = \frac{\mathrm{i}}{\pi} \int_0^\infty \sqrt{t} \, (t - \varepsilon^2 \, L_+^2)^{-1} \, M_k \, (t - \varepsilon^2 \, L_-^2)^{-1} \, \mathrm{d}t \tag{18a}$$

where $M_0 = M$ and, for $k \geq 1$,

$$M_k = - \sum_{j+l=k-1} F_j \, F_l \,. \tag{18b}$$

Moreover, we find that (18) defines a family of operators which is uniformly bounded with respect to ε on H^s. This property is the essential prerequisite for our main result in Section 4.

3. The operator equation $AX + XB = Y$

In this section we will establish an integral representation for the solution of the operator equation

$$AX + XB = Y \,, \tag{19}$$

known as *Sylvester's equation*, on $H^s(\mathbb{T})$. The formal construction follows the approach of [1] for a generalized matrix-Sylvester equation in finite dimensions. Here, however, we face additional difficulties due to the fact that our A and B are unbounded operators on a Hilbert space.

We begin by reviewing some facts about operators on an abstract Hilbert space $\mathcal{H}$. We write $\mathcal{L}(\mathcal{H})$ to denote the space of bounded operators on $\mathcal{H}$. For $0 < r < 1$, we define

$$\Omega_r = \{z \in \mathbb{C}\backslash\{0\} \colon -r\pi < \operatorname{Arg} z < r\pi\} \,. \tag{20}$$

A possibly unbounded linear operator $E \colon D(E) \subset \mathcal{H} \to \mathcal{H}$ is *strictly accretive* if there exists $\alpha > 0$ such that

$$\operatorname{Re}\langle Ex, x\rangle \geq \alpha \, \|x\|^2 \tag{21}$$

for all $x \in D(E)$; see, e.g., [6].

Proposition 1. *If $A \in \mathcal{L}(\mathcal{H})$ is normal with $\sigma(A) \subset \Omega_{\frac{1}{2}}$, then A is strictly accretive. Moreover, there exist $\delta > 0$ such that for all $E \in \mathcal{L}(\mathcal{H})$ with $\|A - E\| < \delta$, E is also strictly accretive.*

Proof. Since A is a normal bounded operator, the closure of its numerical range

$$W(A) = \{\langle Ax, x\rangle \colon x \in \mathcal{H}, \|x\| = 1\} \tag{22}$$

is the smallest convex set containing $\sigma(A)$; see, e.g., [4]. As $\sigma(A)$ itself is closed and contained in $\Omega_{\frac{1}{2}}$, $\operatorname{dist}(W(A), \mathrm{i}\mathbb{R}) > 0$. In other words, there exists $\alpha > 0$ such that

$$\operatorname{Re}\langle Ax, x\rangle \geq \alpha \, \|x\|^2 \tag{23}$$

for all $x \in \mathcal{H}$, i.e., A is strictly accretive. Now suppose that there exist sequences $E_n \in \mathcal{L}(\mathcal{H})$ and $x_n \in \mathcal{H}$ with $\|E_n - A\| < \frac{1}{n}$ and $\|x_n\| = 1$ such that

$$\operatorname{Re}\langle E_n x_n, x_n\rangle < \frac{\alpha}{2} \,. \tag{24}$$

Then

$$\operatorname{Re}\langle (E_n - A) x_n, x_n\rangle + \operatorname{Re}\langle A x_n, x_n\rangle < \frac{\alpha}{2} \,, \tag{25}$$

so that

$$\liminf_{n\to\infty} \operatorname{Re}\langle A x_n, x_n\rangle \leq \frac{\alpha}{2} \,, \tag{26}$$

which contradicts (23). □

Theorem 2 (Kato [6])**.** *Let $E \colon D(E) \subset \mathcal{H} \to \mathcal{H}$ be strictly accretive. Then there exists a unique strictly accretive operator $E^{\frac{1}{2}}$ defined on $D(E)$ such that $(E^{\frac{1}{2}})^2 = E$ and*

$$E^{\frac{1}{2}} u = \frac{1}{\pi} \int_0^\infty t^{-\frac{1}{2}} \, E \, (t + E)^{-1} \, u \, \mathrm{d}t \tag{27}$$

for all $u \in D(E)$.

Within this framework, we can prove a first solution theorem for Sylvester's equation on a Hilbert space, at this point for bounded normal operators A and B.

Theorem 3. *Let $A, B, Y \in \mathcal{L}(\mathcal{H})$ such that A and B are normal with $\sigma(A) \subset \Omega_{\frac{1}{4}}$ and $\sigma(B) \subset \operatorname{cl} \Omega_{\frac{1}{4}}$. Then*

$$X = \frac{1}{\pi} \int_0^\infty t^{\frac{1}{2}} \, (t + A^2)^{-1} \, Y \, (t + B^2)^{-1} \, \mathrm{d}t \tag{28}$$

defines a bounded operator on $\mathcal{H}$ which solves the operator Sylvester equation

$$AX + XB = Y \, . \tag{29}$$

Remark 4. More generally, Theorem 3 remains true if one of the operators A and B has a spectrum strictly in the interior of $\Omega_{\frac{1}{4}}$. The spectrum of the other operator may, in particular, include zero.

Proof. Following [1], let $\varphi(A) = A^2$. By Theorem 2, its inverse is given by

$$\psi(E) \equiv E^{\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty t^{-\frac{1}{2}} \, E \, (t + E)^{-1} \, \mathrm{d}t \, . \tag{30}$$

Proposition 1 asserts that this expression is well-defined on a neighborhood of every normal opearator E with $\sigma(E) \subset \Omega_{\frac{1}{2}}$. In particular, it is possible to define the Fréchet derivative of (30).

The derivative of φ at A is a linear map on $\mathcal{L}(\mathcal{H})$ whose action is given by

$$D\varphi(A)X = AX + XA \, . \tag{31}$$

Let us first consider the case when $B = A$. In this case, equation (29) takes the form

$$D\varphi(A)X = Y \tag{32}$$

and its solution can be found by inverting $D\varphi(A)$. Noting that $D\varphi(A)^{-1} = D\psi(\varphi(A))$ and differentiating (30) under the integral sign, we obtain

$$D\psi(E)Y = \frac{1}{\pi} \int_0^\infty t^{\frac{1}{2}} \, (t + E)^{-1} \, Y \, (t + E)^{-1} \, \mathrm{d}t \, . \tag{33}$$

(To justify this operation, we note that the Fréchet derivative can be seen as a closed operator. A lemma of Hille [5] asserts that closed operators can be moved inside a Bochner integral.) Thus, setting $E = \varphi(A)$ so that $\sigma(E) \subset \Omega_{\frac{1}{2}}$, we obtain

$$X = D\psi(A^N)Y = \frac{1}{\pi} \int_0^\infty t^{\frac{1}{2}} \, (t + A^2)^{-1} \, Y \, (t + A^2)^{-1} \, \mathrm{d}t \, . \tag{34}$$

We next consider a perturbed version of the general case. Let $\gamma > 0$ and consider the operator equation

$$A \, X_\gamma + X_\gamma \, B_\gamma = Y \tag{35}$$

where the shifted operator $B_\gamma \equiv B + \gamma I$ has its spectrum contained in $\Omega_{\frac{1}{4}}$. We set

$$\tilde{A}_\gamma = \begin{pmatrix} A & 0 \\ 0 & B_\gamma \end{pmatrix} , \quad \tilde{Y} = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{X}_\gamma = \begin{pmatrix} 0 & X_\gamma \\ 0 & 0 \end{pmatrix} . \tag{36}$$

Solving

$$\tilde{A}_\gamma \, \tilde{X}_\gamma + \tilde{X}_\gamma \, \tilde{A}_\gamma = \tilde{Y} \tag{37}$$

on $\mathcal{H} \times \mathcal{H}$ is equivalent to solving (35) on $\mathcal{H}$. Since $\sigma(\tilde{A}) = \sigma(A) \cup \sigma(B_\gamma) \subset \Omega_{\frac{1}{4}}$, we can apply the result from the previous step: The solution $\tilde{X}_\gamma$ is given by

$$\tilde{X}_\gamma = \frac{1}{\pi} \int_0^\infty t^{\frac{1}{2}} \, (t + \tilde{A}_\gamma^2)^{-1} \, \tilde{Y} \, (t + \tilde{A}_\gamma^2)^{-1} \, \mathrm{d}t \tag{38}$$

so that

$$X_\gamma = \frac{1}{\pi} \int_0^\infty t^{\frac{1}{2}}\, (t + A^2)^{-1}\, Y\, (t + B_\gamma^2)^{-1}\, \mathrm{d}t\,. \tag{39}$$

We finally consider the limit $\gamma \to 0$. Since

$$\| t^{\frac{1}{2}}\, (t + A^2)^{-1}\, Y\, (t + B_\gamma^2)^{-1} \| \leq t^{\frac{1}{2}}\, \frac{1}{t + \alpha}\, \frac{1}{t}\, \|Y\|\,, \tag{40}$$

where $\alpha > 0$ is a lower bound on the spectrum of A^2 so that the right hand side is integrable on $[0, \infty)$, the dominated convergence theorem for Bochner integrals applies (see, e.g., [11]). This proves that $X_\gamma \to X$ in $\mathcal{L}(\mathcal{H})$, where X is given by (28). Moreover,

$$A\, X_\gamma + X_\gamma\, B_\gamma \to AX + XB \tag{41}$$

in $\mathcal{L}(\mathcal{H})$, so that X solves (29). □

We now formulate the solution theorem for the case when A and B are unbounded operators. There are two differences. First, we need to require that Y is bounded on $\mathcal{H}$ and $D(A)$. Second, we need to have well-defined spectral projections of A and B that preserve their resolvent estimates. For simplicity, we state the result in the concrete context where Y is one of the M_k from Section 2 so that the latter consideration is trivial; more general statements are possible.

Proposition 5. *Given $Z \in \mathcal{L}(H^s)$ with $s \in \mathbb{R}$,*

$$X = \frac{\mathrm{i}}{\pi} \int_0^\infty t^{\frac{1}{2}}\, (t - \varepsilon^2 L_+^2)^{-1}\, Z\, (t - \varepsilon^2 L_-^2)^{-1}\, \mathrm{d}t \tag{42}$$

defines a family of linear operators on H^s, uniformly bounded with respect to ε. If, in addition, $Z \in \mathcal{L}(H^{s+1})$, X solves the operator Sylvester equation

$$\varepsilon\, L_+ X - \varepsilon\, X L_- + Z = 0\,. \tag{43}$$

Proof. We set $A = -\varepsilon \mathrm{i} L_+ > 0$, $B = \varepsilon \mathrm{i} L_- \geq 0$, and $Y = \mathrm{i} Z$. Clearly, $A, B \in \mathcal{L}(H^{s+1}, H^s)$; the two operators are self-adjoint, diagonal in the Fourier representation, and satisfy, for $t > 0$, the resolvent estimates

$$\|(t + A^2)^{-1}\|_{\mathcal{L}(H^s)} \leq \frac{1}{1 + t} \quad \text{and} \quad \|(t + B^2)^{-1}\|_{\mathcal{L}(H^s)} \leq \frac{1}{t} \tag{44}$$

independent of s and ε. Hence,

$$\| t^{\frac{1}{2}}\, (t + A^2)^{-1}\, Y\, (t + B^2)^{-1} \|_{\mathcal{L}(H^s)} \leq t^{\frac{1}{2}}\, \frac{1}{1 + t}\, \frac{1}{t}\, \|Y\|_{\mathcal{L}(H^s)}\,, \tag{45}$$

so that (42) defines a family of uniformly bounded linear operators on H^s.

For $k \in \mathbb{N}^*$, let $\mathbb{P}_k$ denote the spectral projector onto $[0, k]$ and set $\mathbb{Q}_k = I - \mathbb{P}_k$. Then the operators $A_k = \mathbb{P}_k A + \mathbb{Q}_k I$ and $B_k = \mathbb{P}_k B + \mathbb{Q}_k I$ satisfy estimates of the form (44) independent of k. Now assuming that, in addition, $Y \in \mathcal{L}(H^{s+1})$,

$$X_k = \frac{1}{\pi} \int_0^\infty t^{\frac{1}{2}}\, (t + A_k^2)^{-1}\, Y\, (t + B_k^2)^{-1}\, \mathrm{d}t \tag{46}$$

is also well-defined and bounded independent of k and ε as a family of linear operators on H^r for $r \in \{s, s+1\}$. Moreover,

$$(t + A_k^2)^{-1} \to (t + A^2)^{-1} \quad \text{and} \quad (t + B_k^2)^{-1} \to (t + B^2)^{-1} \tag{47}$$

in the strong operator topology on H^r, pointwise for $t > 0$. Thus, $X_k \to X$, also in the strong operator topology on H^r. Finally, we know from Theorem 3 that X_k solves

$$A_k X_k + X_k B_k = Y \,. \tag{48}$$

Thus, for $u \in H^{s+1}$, noting that the spectral projectors converge in the strong operator topology, we have

$$\begin{aligned} \|(A_k X_k - AX)u\|_s &= \|A_k\,(X_k - X)u + (A_k - A)\,Xu\|_s \\ &\le \|A_k\|_{\mathcal{L}(H^{s+1},H^s)}\,\|(X_k - X)u\|_{s+1} + \|(A_k - A)Xu\|_s \\ &\to 0 \end{aligned} \tag{49}$$

and

$$\begin{aligned} \|(X_k B_k - XB)u\|_s &= \|X_k(B_k - B)u + (X_k - X)Bu\|_s \\ &\le \|X_k\|_{\mathcal{L}(H^s)}\,\|(B_k - B)u\|_s + \|(X_k - X)Bu\|_s \\ &\to 0\,. \end{aligned} \tag{50}$$

This proves that

$$AX + XB = Y\,, \tag{51}$$

initially as a linear operator with domain H^{s+1} and, extending by continuity, also on H^s. We finally note that (51) is equivalent to (43). □

Corollary 6. *Suppose $M \in \mathcal{L}(H^s)$ for some $s \in \mathbb{R}$. Then each of the F_k, recursively defined in* (18)*, is bounded in H^s independent of ε. If, in addition, $M \in \mathcal{L}(H^{s+1})$, F_k satisfies the order condition* (17)*.*

Proof. Each of the order conditions is in the form of the Sylvester equation (43) with M_k in place of Z. Thus, the claim is asserted by Proposition 5 provided the M_k are bounded operators on H^s and H^{s+1}. For $k = 0$ this holds by assumption. For $k \ge 1$, we note that the recursion (18b) maintains this property. □

4. Main theorem

Theorem 7. *Suppose $M \in \mathcal{L}(H^s) \cap \mathcal{L}(H^{s+1})$ and $u_0 \in H^{s+1}(\mathbb{T})$ for some $s \in \mathbb{R}$. Fix $N \in \mathbb{N}_0$ and let F^{N+1}_{slow} be defined by the recursion* (18)*. Then there exist a unique solution $u \in C([0,\infty), H^{s+1})$ to the Klein–Gordon equation* (4) *with prepared initial data $u(0) = u_0$ and $v(0) = L_- u_0 + F^N_{\text{slow}} u_0$ and a unique solution $U \in C([0,\infty), H^{s+1})$ to the slow equation* (15) *with initial data $U(0) = u_0$. These solutions are bounded in $L^\infty([0,T]; H^{s+1})$ for any fixed $T > 0$ independent of ε.*

Moreover, for every $T > 0$ there is a constant c indepent of ε such that

$$\sup_{0 \le t \le T} \|u(t) - U(t)\|_s \le c\,\varepsilon^{N+1}\,. \tag{52}$$

Proof. Since Corollary 6 asserts that $F^N_{\text{slow}} \in \mathcal{L}(H^{s+1})$ with uniform bounds in ε, existence and uniqueness of solutions follow by standard semigroup theory. Moreover, again by Corollary 6, the F_k satisfy the order conditions (17) so that, continuing the argument from (14),

$$\partial_t w = \left(\frac{\mathrm{i}}{\varepsilon} - L_- - F^{N+1}_{\text{slow}}\right) w + \varepsilon^{N+1}\,R_N u \tag{53}$$

with remainder operator $R_N \in \mathcal{L}(H^s)$ given by

$$R_N = -\sum_{k=N+1}^{2N+2} \varepsilon^{k-N-1} \sum_{j+l=k} F_j\, F_l\,. \tag{54}$$

Taking the H^s inner product of (53) with w, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_s^2 = \operatorname{Re}\langle F_{\text{slow}}^{N+1} w + \varepsilon^{N+1}\, R_N u, w\rangle \leq c_1\, \|w\|_s^2 + c_2\, \|u\|_s\, \|w\|\, \varepsilon^{2N+2}\,. \tag{55}$$

We note that the initial condition ensures that $w(0) = O(\varepsilon^{N+1})$, so that w will remain $O(\varepsilon^{N+1})$ in H^s for any finite interval of time. Finally, taking the difference between (14a) and (15), we obtain

$$\partial_t(u - U) = L_-(u - U) + F_{\text{slow}}^N(u - U) + w + \varepsilon^{N+1}\, F_{N+1} u\,. \tag{56}$$

As L_- generates a unitary semigroup, a standard energy estimate completes the proof. □

Remark 8. In the typical case where M is defined by multiplication with a potential $\phi \in H^s$, the assumptions of the theorem are satisfied for any $s > \frac{1}{2}$ due to the Sobolev algebra property of these spaces.

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