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HAMILTONIAN FORMALISM FOR MODELS OF ROTATING SHALLOW WATER IN SEMIGEOSTROPHIC SCALING

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Abstract. This paper presents a first rigorous study of the so-called large-scale semigeostrophic equations which were first introduced by R. Salmon in 1985 and later generalized by the first author. We show that these models are Hamiltonian on the group of H^s diffeomorphisms for $s > 2$. Notably, in the Hamiltonian setting an apparent topological restriction on the Coriolis parameter disappears. We then derive the corresponding Hamiltonian formulation in Eulerian variables via Poisson reduction and give a simple argument for the existence of H^s solutions locally in time.

1. **Introduction.** The study of so-called *balance models* which describe approximate slow manifolds in Hamiltonian systems with strong gyroscopic forces in the limit of vanishing inertia is a recurring theme in geophysical fluid dynamics. This regime is known as the semi-geostrophic limit, the small parameter is known as the Rossby number [22]. It is typical for mid-latitude large-scales flows in atmosphere and ocean, particularly for the dynamics of strong fronts.

It is well known that the equations of motion of the parent dynamics, in the simplest and most typical case the rotating shallow water equations, can be derived as the Euler–Lagrange equations from a Hamilton principle. We study approximate equations for the dynamics on the slow time scale which, formally, preserve this structure. One classical example of such structure preserving models for the slow dynamics are the *semigeostrophic equations* [10, 11] which, using a transformation proposed by Hoskins [12], can be transformed into a coordinate system in which they can be solved by simple advection of the scalar *potential vorticity* coupled with a nonlinear elliptic equation for the velocity in terms of the potential vorticity. The Hoskins transformation can be interpreted as a Legendre transform; this observation is the key to the proof of well-posedness [4, 7].

While the derivation of the semigeostrophic equations as a structure-preserving approximation was historically incidental, Salmon [20, 21] pioneered the point of view that such reduced models can be derived systematically by performing all approximations on the variational principle. If the symmetries of the Lagrangian—here the time translation invariance and the particle relabeling symmetry—are preserved, the resulting equations of motion will have analogs of the appropriate

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physical conservation laws—here the conservation of energy and the advection of potential vorticity.

In his 1985 paper [20], Salmon suggested two models for semigeostrophic flow, the so-called L_1 and the LSG model. The former is obtained by constraining the shallow water Hamilton principle to geostrophic balance, the latter differs by a near-identity change of coordinates with higher order terms dropped. While the L_1 model is well posed (in fact, its local well-posedness is a special case of our results in Section 7), the LSG model, though formally distinguished by its canonical symplectic structure, appears to be ill posed.

Based on the observation that all these existing reduced models have an affine Lagrangian (a Lagrangian which is linear in the velocities and thus degenerate), a reinterpretation of Salmon's method was proposed in [16]. The two steps of Salmon's procedure—constraining the phase space followed by an approximate near-identity change of coordinates—can be reversed. Namely, a natural *first* step is to choose a near-identity change into new coordinates such that the perturbation expansion in the Rossby number of the transformed Lagrangian degenerates when consistently truncated to a chosen order. A degenerate Lagrangian will subsequently imply so-called Dirac constraints. Thus, the constraints arise as a consequence of the dropping of higher order terms, not as an *a priori* assumption.

This new point of view has three advantages. First, there is only one approximation step—the truncation of a single asymptotic series—which is conceptionally cleaner and potentially more amenable to rigorous analysis. Second, it is more general. There are entire classes of near-identity transformations which render the truncated transformed Lagrangian degenerate. In particular, there are distinguished cases which have not been noted before. Third, all classical nearly geostrophic models can be expressed in this framework and can therefore be consistently compared.

As in [20], we use the rotating shallow water equations in semigeostrophic scaling as a model setting. The derivation of first order approximate models via the transformational approach is carried out in [16] for constant Coriolis parameter f and in [17] for a spatially varying Coriolis parameter. We call the resulting models *generalized LSG equations* (GLSG). They can be formulated as an advection equation

$$\partial_t q + u \cdot \nabla q = 0 \tag{1a}$$

for the *potential vorticity*

$$q = \frac{f + \varepsilon\,(\lambda + \frac{1}{2})\,\Delta h}{h}\,. \tag{1b}$$

The system is closed via a relation which expresses the advecting velocity field u in terms of the *layer depth* h,

$$\bigl[f - \varepsilon\,(\lambda + \tfrac{1}{2})\,(h\,\Delta + 2\,\nabla h \cdot \nabla)\bigr] u = \nabla^\perp \bigl[h - \varepsilon\,\lambda\,(2\,h\,\Delta h + |\nabla h|^2)\bigr]\,. \tag{1c}$$

In the above, λ is a free parameter which determines the choice of coordinates for the balance model. There are several distinct values: $\lambda = \frac{1}{2}$ yields Salmon's L_1 dynamics, $\lambda = -\frac{1}{2}$ yields Salmon's LSG equations, both proposed in [20]. The remaining choices of λ are unexplored. We remark that when $\lambda = 0$, by an elliptic regularity argument, the q-u inversion gains three derivatives; it gains one derivative at best for all other choices of λ.

We note that the generalized LSG equations are different from the classical semigeostrophic equations [12], although they apply in the same asymptotic limit. Despite the remarkable structure of the latter [4, 7], there are at least two reasons to look at alternatives. First, the semigeostrophic equations become unwieldy for non-constant Coriolis parameter (structure persists in so-called vorticity coordinates, but explicit formulas and invertibility criteria are hard to come by, see the discussion in [19, 17]). Further, second order semigeostrophic models are hard to derive and we do not know about any structural or analytically rigorous results. The generalized LSG framework appears to be principally suited for a systematic study of higher order models, although they get complicated so quickly that this possibility may not be of practical concern [16].

All the results quoted above are purely formal. In this paper, we make a first step toward a rigorous understanding of the generalized LSG equations. First, we present a geometric framework for the reduced system. A typical fluid dynamical system is Hamiltonian on the cotangent bundle of a diffeomorphism group. In the reduced case, the phase space is the group of H^s diffeomorphisms itself. Though this is not particularly surprising in the light of the observed reduction in the degrees of freedom of the system, we are then able to give sufficient conditions which ensure that the symplectic form is non-degenerate.

Our results, in particular, clarify the following issue that was left open in [16]. The Lagrangian formulation of the rotating shallow water equations as well as the derivation of nearly geostrophic approximations via variational asymptotics necessarily require that the Coriolis parameter is exact (i.e., that it can be written as the curl of a vector potential). We show here that the Hamiltonian form of the equations is valid, however, without unphysical topological restrictions on the Coriolis parameter. The resolution is based on the observation that GLSG symplectic form on the torus does not arise from variational principle as a pullback of the canonical symplectic form via Legendre transform. Instead, it is a sum of two closed two-forms subordinate to the decomposition of the Coriolis parameter into zero-mean and constant parts, respectively. The first of these summands is, in fact, a pull-back form, while the other is a symplectic form specific to the diffeomorphism group of the two-torus.

The paper is organized as follows. Section 2 reviews notation and basic concepts from the theory of diffeomorphism groups and how it applies to fluid flow. Section 3 introduces the class of models we are studying and sketches their derivation. In Section 4, we derive the Hamiltonian formulation of these models in the Lagrangian representation. In Section 5, we derive the corresponding Hamiltonian formulation in Eulerian variables via Poisson reduction.

All the above results presume that solutions in Sobolev spaces H^s with $s > 2$ exist. Thus, in the final Sections 6 and 7 we provide a simple argument that our family of models indeed possesses such solutions locally in time provided that the initial potential vorticity is a sufficiently small H^s-perturbation of unity. Consequently, our family of models is not only formally Hamiltonian, but is also well-posed as a Cauchy problem. This result contains, as a special case, the first proof of local well-posedness for Salmon's L_1 model [20] that we know of. In a subsequent paper [5], we shall show by using more subtle, but also more involved estimates, that we can extend the existence of solutions globally in time under the weaker and physically motivated restriction that the initial potential vorticity is strictly positive.

2. **Diffeomorphism groups.**

Diffeomorphism groups arise naturally as configuration manifolds in various fluid models. In this section, we fix notation and recall the necessary known facts from the theory of diffeomorphism groups following [9, 18, 8].

Let D denote a compact connected oriented two-dimensional Riemannian manifold without boundary, and let $H^s(D,N)$ denote the space of mappings of Sobolev class s from D into another manifold without boundary N; we abbreviate $H^s(D,D) \equiv H^s(D)$ and use $\|\cdot\|_s$ to denote the H^s norm. Similarly, let $C^k(D,N)$ denote the space of k times continuously differentiable maps from D to N, again with $C^k(D,D) \equiv C^k(D)$. The Sobolev embedding theorem ensures that there is a dense continuous inclusion $H^s(D) \subset C^k(D)$ for $s > 1+k$. Finally, we write J_η to denote the Jacobian of the map η.

For $s > 2$, we define the *diffeomorphism group*

$$\mathcal{D}^s = \{\eta \in H^s(D) \mid \eta \text{ is bijective and } J_\eta \neq 0\} \,. \tag{2}$$

It is well known that that $\mathcal{D}^s$ is a smooth infinite dimensional manifold [9]. It is also a topological group with the group operation being the composition of maps. For fixed $\xi \in \mathcal{D}^s$, composition on the right, i.e. $\eta \mapsto \eta \circ \xi$, is a smooth map, while composition on the left, i.e. $\eta \mapsto \xi \circ \eta$ and inversion $\eta \mapsto \eta^{-1}$ are merely continuous as maps from $\mathcal{D}^s$ to itself and C^k as maps from $\mathcal{D}^{s+k}$ to $\mathcal{D}^s$. The group of volume preserving diffeomorphisms,

$$\mathcal{D}^s_{\text{vol}} = \{\eta \in \mathcal{D}^s \mid J_\eta = 1\} \,, \tag{3}$$

is a closed submanifold and a topological subgroup of $\mathcal{D}^s$.

The tangent space $T_\eta\mathcal{D}^s$ at η is the space $\mathfrak{X}^s(D)$ of H^s vector fields over η; the tangent space $T_\eta\mathcal{D}^s_{\text{vol}}$ at η is the corresponding subspace of divergence free vector fields.

When $s > 1$ and and η_t is a C^1 curve in $\mathcal{D}^s$ through the identity, it is the flow of the possibly time dependent vector field u via

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\eta_t = u \circ \eta_t \,. \tag{4}$$

In the general case, i.e. when η_0 is not necessarily the identity map, we call η_t satisfying (4) a *shifted flow* of u. Conversely, if u is a time-continuous H^s vector field on D and $s > n/2+2$, then the flow η_t of u is a C^1 curve in $\mathcal{D}^s$ with $\eta_0 = \text{id}$ and (4) holds.

For $u \in \mathfrak{X}^s(D)$, define $\hat{u} \colon \mathcal{D}^s \to T\mathcal{D}^s$ via $\eta \mapsto u \circ \eta$. Then

$$[\hat{u}, \hat{v}] = \widehat{[u,v]} \,, \tag{5}$$

where the left hand bracket denotes the Lie bracket of vector fields on $\mathcal{D}^s$ while the right hand bracket denotes the Lie bracket of vector fields on D.

Finally, let $h \colon \mathcal{D}^s \to H^{s-1}(D,\mathbb{R})$ be defined via $h(\eta) = J_{\eta^{-1}}$, the Jacobian of the inverse of η. When η is a flow of u, possibly shifted, the transport theorem and the change of coordinates formula imply that h satisfies the *continuity equation*

$$\partial_t h + \operatorname{div}(hu) = 0 \,. \tag{6}$$

Here and in the remainder of this paper, we make a number of purely notational simplifications. When no confusion can occur, we suppress the reference point and write h in place of $h(\eta)$. Moreover, we will sometimes suppress the t-subscript, identifying $\eta(x,t) \equiv \eta_t(x)$. For further reference, we note that the change of variables formula then reads

$$\int_D \phi \circ \eta = \int_D \phi \, h \tag{7}$$

for any integrable function ϕ on D.

3. **Models of semigeostrophic shallow water.** We are now in a position to sketch the derivation of the LSG-type models for semigeostrophic shallow water as introduced in [16]. We assume that D is a flat two-dimensional manifold or a domain in $\mathbb{R}^2$; everything in this section is entirely formal.

Our starting point are the two-dimensional shallow water equation which can be written in semigeostrophic scaling as

$$\varepsilon\,(\partial_t + u\cdot\nabla)u + f\,u^\perp + \nabla h = 0\,, \tag{8a}$$

$$\partial_t h + \operatorname{div}(uh) = 0\,, \tag{8b}$$

where u is the two-dimensional vector of horizontal fluid velocity, h is a fluid depth, $u^\perp = (-u_2, u_1)$, f is the scaled Coriolis parameter, and ε the Rossby number which is assumed to be small.

We first remark that the classical semigeostrophic equations arise via the so-called geostrophic momentum approximation, where the advected, but not the advecting velocities in (8) are replaced by their geostrophic values. We use a different construction, first proposed in [16]. Let R denote a vector potential for the Coriolis parameter f, i.e. $f = \operatorname{curl} R$. Then (8a) arises as the Euler–Lagrange equation of the Lagrangian

$$L_\varepsilon(u\circ\eta) = \int_D \bigl[R\cdot u + \tfrac12\,\varepsilon\,|u|^2 - \tfrac12\,h\bigr]\circ\eta\,, \tag{9}$$

defined on the tangent bundle of the diffeomorphism group of D, where $h(\eta) = J_{\eta^{-1}}$. We favor the more compact notation (9) over the traditional $L_\varepsilon(\eta,\dot\eta)$ with u and $\dot\eta$ related by (4) throughout this paper. We apply a near-identity change of configuration variables which, to first order in ε, is generated by

$$v = \tfrac12\,u^\perp + \lambda\,\nabla h\,. \tag{10}$$

Then, dropping terms of order ε^2 and higher in (9), we obtained the one-parameter family of affine Lagrangians

$$L_\lambda(u\circ\eta) = \int_D (R\cdot u - \tfrac12\,h)\circ\eta + \varepsilon\int_D \bigl[(\lambda+\tfrac12)\,\nabla^\perp h\cdot u - \lambda\,|\nabla h|^2\bigr]\circ\eta \tag{11}$$

The corresponding Euler–Lagrange equations are then found to be precisely the u-h relation (1c). Equations (1c) and (8b) comprise the GLSG system. Since the Lagrangian (11) is invariant under particle relabellings, the Noether theorem yields the potential vorticity (1b) as a Lagrangian invariant. We conclude that equations (1a–c) form a closed system for the reduced dynamics, which provides a formally equivalent formulation of the GLSG equations. In Section 7 we shall show that the GLSG equations are indeed well posed as an initial value problem.

4. **Hamiltonian formalism in Lagrangian representation.** Let us first recall the following construction [14]. Given a Lagrangian $L\colon TQ\to\mathbb{R}$ on a manifold Q with tangent bundle TQ, its *Legendre transform* $FL\colon TQ\to T^*Q$ is the fiber and base point preserving map given by

$$\langle FL(v), u\rangle = \frac{\mathrm{d}}{\mathrm{d}t}_{t=0} L(v+tu) \qquad \text{for all } u,v\in T_qQ\,, \tag{12}$$

where $\langle\cdot,\cdot\rangle$ denotes pairing between T^*Q and TQ. Provided FL is a diffeomorphism, we can define the symplectic form

$$\Omega_L = FL^*(\Omega) = -\mathrm{d}\,FL^*\,\Theta \tag{13}$$

on TQ as the pullback of the canonical symplectic form $\Omega \equiv -\mathrm{d}\Theta$ on T^*Q, where Θ denotes the canonical 1-form on T^*Q. Then, a vector field X_E is Hamiltonian with energy $E = \langle FL(v), v\rangle - L$ with respect to the symplectic form Ω_L if and only if its integral curves satisfy the Euler–Lagrange equations for L.

This standard construction does not apply here as the LSG Lagrangians (11) are affine in u so that the Legendre transforms they define are constant on each fiber. Indeed, direct computation of the exact form Ω_{L_λ} via (13) confirms its degeneracy; see Proposition 2 below. In other words, the tangent bundle of the diffeomorphism group is too large as a phase space for the Hamiltonian formulation of the reduced model equations.

The formal structure of the reduced system explained in Section 3 suggests the diffeomorphism group itself as a candidate phase space. Thus, we must pull back the two-form Ω_{L_λ} and the energy E to the diffeomorphism group by the inclusion of the diffeomorphism group into the zero section of its tangent bundle, in the following denoted i. By direct computation, the pullback $\omega_{L_\lambda} = i^*\Omega_{L_\lambda}$ is an exact two-form on $\mathcal{D}^s$ given by

$$\omega_{L_\lambda}(\eta)(\hat{u}, \hat{w}) = \int_D \bigl[w^\perp \cdot \bigl(\operatorname{curl} R - \varepsilon\,(\lambda + \tfrac{1}{2})(h\,\Delta + 2\,\nabla h \cdot \nabla)\bigr)u\bigr] \circ \eta\,. \tag{14}$$

Similarly, setting $H = i^*E$, we obtain an expression for the energy,

$$H(\eta) = \tfrac{1}{2} \int_D h^2 + 2\varepsilon\lambda\, h\, |\nabla h|^2\,. \tag{15}$$

The necessary computations are subtle and will be detailed below.

There is, however, a second problem which is topological. On a compact orientable manifold D without boundary, a vector potential of f exists only if f has zero mean. The physical requirement for the validity of nearly geostrophic approximations, on the other hand, is that the Coriolis parameter f has no zeros. To resolve this incompatibility, we notice that, while the vector potential R appears explicitly in the Lagrangian (11), ω_{L_λ} can be written without referencing the vector potential; in this case, however, we must must verify the closedness of the resulting two-form explicitly.

This idea is implemented as follows. On a compact connected orientable manifold without boundary, an arbitrary $f \in C^1(D, \mathbb{R})$ can be written $f = C + \operatorname{curl} R$ for some constant C and vector field R. In Proposition 1 below we verify by direct computation that the two-form

$$\omega_C(\eta)(\hat{u}, \hat{w}) = C \int_D \bigl[w^\perp \cdot u\bigr] \circ \eta \tag{16}$$

is closed. Hence, $\omega_{f,\lambda} = \omega_C + \omega_{L_\lambda}$ is closed. As we shall show in Proposition 5, under certain physically reasonable conditions, $\omega_{f,\lambda}$ is non-degenerate on an open neighborhood of the group of volume preserving diffeomorphisms in the full diffeomorphism group, hence it is a symplectic form on that neighborhood and we find that the the weak form of the u-h inversion (1c) reads

$$\omega_{f,\lambda}(\hat{u}, \hat{w}) = \mathrm{d}H \cdot \hat{w} \tag{17}$$

for all H^s vector fields w on D. This exposes the Hamiltonian structure.

We remark that equation (17) is understood in the sense of an infinite dimensional Hamiltonian systems [6]. In particular, the Hamiltonian vector field $X_H(\eta) = u \circ \eta$ defined by (17) is unbounded in the H^s topology. To proceed, we assume that its integral curves are differentiable as maps $H^{s+k} \mapsto H^s$ for some $k > 0$, postponing the existence issue until Section 7.

We present the details of the argument under the assumption that D is a double periodic domain. This assumption simplifies calculations, but is not essential to the problem. The same technique leads to the results for an arbitrary compact connected orientable two dimensional manifold D provided one replaces differential operations in Cartesian coordinates with their invariant analogues. We further assume that $s > 2$.

Proposition 1. *The form ω_C defined by (16) is a closed two-form on $\mathcal{D}^s$.*

Proof. As $u \cdot v^\perp = -u^\perp \cdot v$, ω_C is skew, thus a two-form. Without loss of generality we may take $C = 1$. Let u, v, and w be arbitrary H^s vector fields on D. Then,

$$\begin{aligned} \mathrm{d}\omega_1(\hat{u}, \hat{v}, \hat{w}) = \tfrac{1}{3} \bigl(& \hat{u} \cdot \omega_1(\hat{v}, \hat{w}) - \hat{v} \cdot \omega_1(\hat{u}, \hat{w}) + \hat{w} \cdot \omega_1(\hat{u}, \hat{v}) \\ & - \omega_1([\hat{v}, \hat{w}], \hat{u}) + \omega_1([\hat{u}, \hat{w}], \hat{v}) - \omega_1([\hat{u}, \hat{v}], \hat{w}) \bigr) , \end{aligned} \tag{18}$$

where $\hat{u} \cdot \omega_1(\hat{v}, \hat{w})$ denotes differentiation of the function $\omega_1(\hat{v}, \hat{w})$ by the vector field $\hat{u}$.

Now fix $\eta \in \mathcal{D}^s$ and let η_t be an integral curve of $\hat{u}$ through η, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}_{t=0} \eta_t = \hat{u}(\eta) = u \circ \eta$. Then, using the change of variables formula (7) and the continuity equation (6), we obtain

$$\begin{aligned} (\hat{u} \cdot \omega_1(\hat{v}, \hat{w}))(\eta) &= \frac{\mathrm{d}}{\mathrm{d}t}_{t=0} \omega_1(\eta_t)(\hat{v}, \hat{w}) = \frac{\mathrm{d}}{\mathrm{d}t}_{t=0} \int_D (v \cdot w^\perp) \, h_t \\ &= - \int_D (v \cdot w^\perp) \, \operatorname{div}(hu) = \int_D h \, (w^\perp \cdot \nabla_u v - v^\perp \cdot \nabla_u w) \, . \end{aligned} \tag{19}$$

Recall that, due to (5), for arbitrary vector fields u, v on D,

$$[\hat{u}, \hat{v}] = \widehat{[u, v]} = (\nabla_u v - \nabla_v u) \circ \eta \, . \tag{20}$$

Substituting identities (19) and (20) with corresponding permutations of their arguments into (18), we readily obtain that $\mathrm{d}\omega_1(\eta)(\hat{u}, \hat{v}, \hat{w}) = 0$. $\square$

Let $L_\lambda \colon T\mathcal{D}^s \to \mathbb{R}$ be the Lagrangian given by formula (11). Then, the definition of the Legendre transform (12) directly yields

$$\langle FL_\lambda(v \circ \eta), u \circ \eta \rangle = \int_D (R \cdot u) \circ \eta + \varepsilon \, (\lambda + \tfrac{1}{2}) \int_D (\nabla^\perp h \cdot u) \circ \eta \, . \tag{21}$$

Note that, as L_λ is affine, the right hand side of (21) does not depend on v.

Proposition 2. *Let Ω be the canonical symplectic form on $T^*\mathcal{D}^s$. Then its pullback to $T\mathcal{D}^s$, $\Omega_{L_\lambda} = FL_\lambda^* \, \Omega$, is given by*

$$\Omega_{L_\lambda}(v \circ \eta)(U, W) = \int_D \bigl(\bigl[\operatorname{curl} R - \varepsilon \, (\lambda + \tfrac{1}{2}) \, (h \, \Delta + 2 \, \nabla h \cdot \nabla) \bigr] u_1 \cdot w_1^\perp \bigr) \circ \eta \, , \tag{22}$$

where v is an H^s vector field on D; $U, W \in T_{v \circ \eta} T\mathcal{D}^s$ are arbitrary; $u_1 \circ \eta = T\tau U$ and $w_1 \circ \eta = T\tau W$ with $\tau \colon T\mathcal{D}^s \to \mathcal{D}^s$ denoting the canonical projection and T denoting the tangent map.

Proof. As is customary, we identify the tangent space $T_\eta \mathcal{D}^s$ with a dense subset of the cotangent space $T_\eta^* \mathcal{D}^s$ via the L^2 metric

$$\langle u \circ \eta, v \circ \eta \rangle = \int_D (u \cdot v) \circ \eta \, , \tag{23}$$

where u and v are arbitrary H^s vector field on D and $\langle \cdot, \cdot \rangle$ denotes a pairing between $T^*\mathcal{D}^s$ and $T\mathcal{D}^s$. With such an identification in place, formula (21) reads

$$FL_\lambda(v \circ \eta) = \big(R + \varepsilon\,(\lambda + \tfrac{1}{2})\,\nabla^\perp h\big) \circ \eta\,. \tag{24}$$

The rest is a long calculation via the usual formula for pullbacks,

$$\phi^*\Omega(q)(U, W) = \Omega(\phi(q))(T\phi \cdot U, T\phi \cdot W)\,, \tag{25}$$

with $q = v \circ \eta$ and $\phi = FL_\lambda$. Let $v_t \circ \eta_t$ be a curve in $T\mathcal{D}^s$ through $v \circ \eta$ with velocity U. Then, $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \eta_t = u_1 \circ \eta$. Calculating in a chart about η, using (24) and the continuity equation (6),

$$\begin{aligned} T\,FL_\lambda(v \circ \eta)U &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} FL_\lambda(v_t \circ \eta_t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(R + \varepsilon\,(\lambda + \tfrac{1}{2})\nabla^\perp h_t)\big) \circ \eta_t \\ &= \Big(u_1, u_1 \cdot \nabla R + \varepsilon\,(\lambda + \tfrac{1}{2})\,\big[u_1 \cdot \nabla\nabla^\perp h - \nabla^\perp \operatorname{div}(h u_1)\big]\Big) \circ \eta\,. \end{aligned} \tag{26}$$

The canonical symplectic form Ω on $T^*\mathcal{D}^s$ is given in charts by

$$\Omega(\alpha_\eta)((u \circ \eta, \beta_1), (w \circ \eta, \beta_2)) = \langle \beta_2, u \circ \eta \rangle - \langle \beta_1, w \circ \eta \rangle\,, \tag{27}$$

where $\alpha_\eta, \beta_1, \beta_2 \in T^*_\eta \mathcal{D}^s$ and $u \circ \eta, w \circ \eta \in T_\eta \mathcal{D}^s$. Combining (24), (26), and (27), we obtain

$$\begin{aligned} \Omega_{L_\lambda}(v \circ \eta)(U, W) &= \int_D \big(u_1 \cdot (w_1 \cdot \nabla R) - w_1 \cdot (u_1 \cdot \nabla R)\big] \circ \eta \\ &+ \varepsilon\,(\lambda + \tfrac{1}{2}) \int_D \big[u_1 \cdot (w_1 \cdot \nabla\nabla^\perp h) - w_1 \cdot (u_1 \cdot \nabla\nabla^\perp h)\big] \circ \eta \\ &+ \varepsilon\,(\lambda + \tfrac{1}{2}) \int_D \big[w_1 \cdot \nabla^\perp \operatorname{div}(h u_1) - u_1 \cdot \nabla^\perp \operatorname{div}(h w_1)\big] \circ \eta\,. \end{aligned} \tag{28}$$

The following identities hold for arbitrary vector fields u, w, and R, and scalar field h:

$$u \cdot (w \cdot \nabla R) - w \cdot (u \cdot \nabla R) = u \cdot w^\perp \operatorname{curl} R\,, \tag{29a}$$

$$\int_D \big(w \cdot \nabla^\perp \operatorname{div}(hu)\big) \circ \eta = -\int_D \big(u \cdot \nabla \operatorname{div}(h w^\perp)\big) \circ \eta\,, \tag{29b}$$

$$\nabla \operatorname{div}(h u^\perp) + \nabla^\perp \operatorname{div}(hu) = \Delta h\, u^\perp + 2\,\nabla h \cdot \nabla u^\perp + h\,\Delta u^\perp\,. \tag{29c}$$

These identities now directly imply the equivalence of (28) and (22). □

Let $i(\eta) = 0 \circ \eta$ be the embedding of the diffeomorphism group $\mathcal{D}^s$ into the zero section of its tangent bundle $T\mathcal{D}^s$.

Proposition 3. *The 2-form $\omega_{L_\lambda} = i^*\Omega_{L_\lambda}$ on $\mathcal{D}^s$ is exact. It is given by*

$$\omega_{L_\lambda}(\hat{u}, \hat{w}) = \int_D \Big(\big[\operatorname{curl} R - \varepsilon\,(\lambda + \tfrac{1}{2})\,(h\,\Delta + 2\,\nabla h \cdot \nabla)\big] u \cdot w^\perp\Big) \circ \eta\,. \tag{30}$$

Proof. Recall that $\tau \colon T\mathcal{D}^s \to \mathcal{D}^s$ denotes the canonical projection, so that $\tau \circ i$ is the identity map on $\mathcal{D}^s$ and, therefore, $T\tau T i(u \circ \eta) = u \circ \eta$. Combining this with (25) and (22), we obtain the claimed expression for ω_{L_λ}. Moreover, since the canonical symplectic form $\Omega = -\mathrm{d}\Theta$ is exact and ω_{L_λ} is its pullback,

$$\omega_{L_\lambda} = i^*\, FL_\lambda^*\, \Omega = -i^*\, FL_\lambda^*\, \mathrm{d}\Theta = -\mathrm{d} i^*\, FL_\lambda^*\, \Theta\,. \tag{31}$$

Thus, ω_{L_λ} is exact. □

Proposition 4. *For all $f \in C^1(D,\mathbb{R})$ and $\lambda \in \mathbb{R}$,*

$$\omega_{f,\lambda}(\hat u, \hat w) = \int_D \bigl(\bigl[f - \varepsilon\,(\lambda + \tfrac12)\,(h\,\Delta + 2\,\nabla h \cdot \nabla) \bigr] u \cdot w^\perp \bigr) \circ \eta \tag{32}$$

is a closed 2-form on $\mathcal{D}^s$. Moreover, if $\int_D f = 0$, it is a pullback of the canonical symplectic form on $T^*\mathcal{D}^s$.

Proof. On a compact connected orientable 2-dimensional manifold without boundary the second de Rahm cohomology group is free, generated by a volume form. Therefore, there exist a constant C and a vector field R such that

$$f = C + \operatorname{curl} R\,. \tag{33}$$

Then, $\omega_{f,\lambda} = \omega_C + \omega_{L_\lambda}$ is closed by Propositions 1 and 3.

Since $\int_D \operatorname{curl} R = 0$, the identity $\int_D f = 0$ forces $C = 0$ in (33). Hence, in that case, $\omega_{f,\lambda}$ is a pullback of canonical symplectic form on $T^*\mathcal{D}^s$ by Proposition 3. □

By Proposition 4, $w_{f,\lambda}$ is always closed. Therefore, it is a symplectic form if and only if it is non-degenerate. The degeneracy of $w_{f,\lambda}$ at η is equivalent to the existence of non-trivial solutions to the linear partial differential equation

$$\Lambda_h u \equiv f\,u - \sigma\,(h\,\Delta u + 2\,\nabla h \cdot \nabla u) = 0\,, \tag{34}$$

where $\sigma = \varepsilon\,(\lambda + \frac12)$. In the following, we give a sufficient condition for the injectivity of Λ_h via a simple variational argument. For each $s \geq 0$, we write $\|\cdot\|_s$ to denote a norm on H^s. The particular choice of norm does not matter as it will change only the constants in the estimates, but we assume that the family is such that the norm of the embedding of H^{s+t} into H^s for $t > 0$ equals one.

Proposition 5. *Suppose $f > 0$ is continuous and $0 \leq \sigma \leq 1$. Then, there exists a constant $C = C(D,f)$ such that for all h with $\|h-1\|_{s-1} < C$, the trivial vector field $u = 0$ is the only solution of (34) in H^1.*

Proof. Let $m = \min_{x\in D} f(x)$. We take the scalar product of (34) with u and integrate over D. After integration by parts, we obtain

$$\int_D f\,|u|^2 - \sigma \int_D u \cdot (\nabla h \cdot \nabla u) + \sigma \int_D h\,|\nabla u|^2 = 0\,. \tag{35}$$

When $\sigma = 0$, we have

$$0 = \int_D f\,|u|^2 \geq m\,\|u\|_0^2\,. \tag{36}$$

Therefore, $u = 0$ almost everywhere while $C > 0$ may be chosen arbitrarily.

Now suppose $\sigma > 0$. Without loss of generality we may assume that $2 < s < 3$. By the Sobolev embedding theorem, the inclusions $H^{s-1} \subset W^{1,p}$, $H^{s-1} \subset L^\infty$, and $H^1 \subset L^q$ are continuous with $p = 2/(3-s)$ and $q = 2/(s-2)$. Thus, writing $h = 1 + \tilde h$ in (35) and rearranging terms, applying the Hölder inequality, recalling that $\sigma \leq 1$, and using the continuity of the above embeddings, we estimate

$$\begin{aligned} 0 &= \frac{1}{\sigma} \int_D f\,|u|^2 + \|\nabla u\|_0^2 - \int_D u \cdot (\nabla \tilde h \cdot \nabla u) + \int_D \tilde h\,|\nabla u|^2 \\ &\geq m\,\|u\|_0^2 + \|\nabla u\|_0^2 - \|\nabla \tilde h\|_{L^p}\,\|u\|_{L^q}\,\|\nabla u\|_0 - \|\tilde h\|_{L^\infty}\,\|\nabla u\|_0^2 \\ &\geq m\,\|u\|_0^2 + \|\nabla u\|_0^2 - C_1\,\|\tilde h\|_{s-1}\,\|u\|_1\,\|\nabla u\|_0 - C_2\,\|\tilde h\|_{s-1}\,\|\nabla u\|_0^2 \end{aligned} \tag{37}$$

with constants C_1 and C_2 supplied by Sobolev's inequalities. Now, clearly there exists some $C = C(C_1, C_2, m)$ such that the right hand side of (37) is nonnegative whenever $\|\tilde h\|_{s-1} \leq C$. □

We remark that this result is not sharp, but it is the simplest and most direct estimate possible. More sophisticated estimates will be provided in the context of an L^∞ theory which is the subject of our forthcoming paper [5].

Theorem 4.1. *Suppose* $f \in C^1(D,\mathbb{R})$*,* $f > 0$*, and* $0 \leq \varepsilon\,(\lambda + \frac{1}{2}) \leq 1$*. Then there exists an open set* O *in the diffeomorphism group* $\mathcal{D}^s$ *such that* $\omega_{f,\lambda}$ *is a weak symplectic form on* O*. Moreover,* $\mathcal{D}^s_{\mathrm{vol}} \subset O$*.*

Proof. By Proposition 4, $\omega_{f,\lambda}$ is an exact 2-form on $\mathcal{D}^s$. Proposition 5 with $\sigma = \varepsilon\,(\lambda + \frac{1}{2})$ implies that $\omega_{f,\lambda}$ is non-degenerate on $O = \{\eta \in \mathcal{D}^s \mid \|h(\eta) - 1\|_{s-1} < C\}$, hence a weak symplectic form on O.

The map $\eta \mapsto J_\eta$ is smooth as a map $\mathcal{D}^s \to H^{s-1}$. Indeed, one easily verifies smoothness in charts since the expression for the Jacobian involves only taking first order derivatives of an H^s function as well as addition and multiplication of H^{s-1} functions, all of which are smooth operations as long as H^{s-1} is a topological algebra.

The map $\eta \mapsto \eta^{-1}$ is continuous on $\mathcal{D}^s$, therefore, $\eta \mapsto h(\eta) - 1$ is continuous as a map $\mathcal{D}^s \to H^{s-1}$. Thus, O is open in $\mathcal{D}^s$. If η is a volume preserving diffeomorphism, $h(\eta) \equiv 1$, hence $\eta \in O$. □

Remark 1. Proposition 5 guarantees, in particular, that the set O in Theorem 4.1 can be chosen independent of σ. Thus, one can study the asymptotic behavior as $\varepsilon \to 0$ in the Hamiltonian framework presented here without shrinking the domains of definitions of the symplectic forms.

Let us now turn to computing the Hamiltonian via the same formalism. First, using expression (21) for the Legendre transform, the action $A \equiv \langle FL(v), v \rangle$ reads

$$A(u \circ \eta) = \int_D (R \cdot u) \circ \eta + \varepsilon\,(\lambda + \tfrac{1}{2}) \int_D (\nabla^\perp h \cdot u) \circ \eta\,, \tag{38}$$

so that

$$E(u \circ \eta) = \tfrac{1}{2} \int_D \bigl(h + 2\varepsilon\lambda\,|\nabla h|^2\bigr) \circ \eta = \tfrac{1}{2} \int_D h^2 + 2\varepsilon\lambda\, h\,|\nabla h|^2\,. \tag{39}$$

Identifying $H(\eta) = i^*(E) = E(i(\eta)) = E(0 \circ \eta)$, we recover (15).

Next, to calculate the differential $\mathrm{d}H$, let η_t be a curve through η with initial velocity $w \circ \eta$ and $h_t = h(\eta_t)$. Then, using the continuity equation (8b), the change of variables formula (7), and the identity $\operatorname{div}(h\nabla h) = h\Delta h + |\nabla h|^2$, we obtain for an arbitrary H^s vector field w,

$$\begin{aligned} \mathrm{d}H(\eta) \cdot (w \circ \eta) &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t}_{t=0} \int_D h_t^2 + 2\varepsilon\lambda\, h_t\,|\nabla h_t|^2 \\ &= -\int_D h \operatorname{div}(wh) + \varepsilon\lambda\,\bigl(\operatorname{div}(wh)\,|\nabla h|^2 + 2h\,\nabla h \cdot \nabla \operatorname{div}(wh)\bigr) \\ &= \int_D \bigl(\nabla^\perp\bigl[h + \varepsilon\lambda\,|\nabla h|^2 - 2\varepsilon\lambda\, \operatorname{div}(h\nabla h)\bigr] \cdot w^\perp\bigr) \circ \eta\,. \end{aligned} \tag{40}$$

Next, we show that an everywhere positive fluid depth of mean one (where the latter can always be achieved by an appropriate choice of scales) can be expressed as a Jacobian of a diffeomorphism of D.

Proposition 6. *Suppose* $h_0 > 0$ *and* $\int_D h_0 = \operatorname{vol}(D)$*, where* $\operatorname{vol}(D)$ *is the area of* D*. Then there exists* η_0 *in* $\mathcal{D}^s$ *such that* $h_0 = J_{\eta_0^{-1}}$*. Furthermore, if* $\|h_0 - 1\|_{s-1} < C$*, then* $\eta_0 \in O$*, where* C *and* O *are defined as in Theorem 4.1 and its proof.*

Proof. Without loss of generality, let $D = \{(x,y) \mid 0 \le x, y < 1\}$ so that $\operatorname{vol}(D) = 1$. Then, setting

$$\xi(x,y) = \left(\int_0^1 \int_0^x h_0(x_1,y) \, \mathrm{d}x_1 \, \mathrm{d}y, \frac{\int_0^y h_0(x,y_1) \, \mathrm{d}y_1}{\int_0^1 h_0(x,y) \, \mathrm{d}y} \right), \tag{41}$$

$\eta_0 = \xi^{-1}$ possesses the stated properties. □

We summarize our findings by stating that the generalized LSG equations coincide with Hamilton's equations on the diffeomorphism group $\mathcal{D}^s$.

Theorem 4.2. *Under the assumptions of Theorem 4.1, let $O \supset \mathcal{D}^s_{\mathrm{vol}}$ be the open neighborhood provided therein. Suppose X_H is a Hamiltonian vector field of (15) on O with respect to the symplectic form (32), $\eta \equiv \eta_t \subset O$ an integral curve of X_H, $h = J_{\eta^{-1}}$, and $\frac{\mathrm{d}}{\mathrm{d}t}\eta = u \circ \eta$. Then h and u are classical solutions of the the generalized LSG equations (1).*

Conversely, suppose u and $h > 0$ are classical solutions of the generalized LSG equations (1) with $u \in C([0,T), \mathfrak{X}^s)$ for some $s > 3$ and $h_t(x) \equiv h(t,x)$ satisfies $\|h_t - 1\|_{s-1} < C$ for all $0 \le t < T$ with C as in Proposition 5. Then there is an integral curve η_t of X_H in O such that $h_t = J_{\eta_t^{-1}}$.

Proof. Let η be an integral curve of X_H. As explained in Section 2, the continuity equation (8b) is satisfied since η is a flow of u. Therefore it suffices to verify that equation (1c) holds.

By definition, X_H is Hamiltonian with respect to the symplectic form $\omega_{f,\lambda}$ if $\omega_{f,\lambda}(\eta)(X_H, \hat{w}) = \mathrm{d}H(\eta) \cdot \hat{w}$. On the other hand, $X_H(\eta) = u \circ \eta$, so that

$$\omega_{f,\lambda}(\eta)(u \circ \eta, \hat{w}) = \mathrm{d}H(\eta) \cdot \hat{w} \tag{42}$$

for all $w \in \mathfrak{X}^s(D)$. Substituting in expressions (32) and (40) for the left hand and right hand sides, respectively, we obtain the u-h inversion equation (1c). Finally, note that (1a–c) are equivalent to (1c) and (8b), as can be shown by direct computation, see [16].

Conversely, suppose that u and h are classical solutions of the generalized LSG equations (1c) and (8b) satisfying the conditions of the theorem. By Proposition 6, there is $\zeta \in \mathcal{D}^s$ such that $h_0 = J_{\zeta^{-1}}$. Recall from Section 2 that u generates a flow ξ_t such that ξ_t is a C^1 curve in $\mathcal{D}^s$ with $\xi_0 = \mathrm{id}$. Set $\eta_t = \xi_t \circ \zeta$ and $\hat{h}_t = J_{\eta_t^{-1}}$. Differentiating η_t, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_t = \frac{\mathrm{d}}{\mathrm{d}t}\xi_t \circ \zeta = u \circ \xi_t \circ \zeta = u \circ \eta_t \,. \tag{43}$$

Thus, η_t is a shifted flow of u and $\hat{h}_t$ satisfies the continuity equation (8b) with the initial condition $\hat{h}_0 = J_{\zeta^{-1}} = h_0$. However, for a given H^s vector field u, the solution of the Cauchy problem for the linear continuity equation (8b) is unique in $L^2(D)$. This follows from a a direct estimate on $\tilde{h} = h - \hat{h}$, namely

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{h}\|^2_{L^2} \le \int_D \operatorname{div}(u\tilde{h}) \, \tilde{h} \le \frac{1}{2} \, \|\operatorname{div} u\|_{L^\infty} \|\tilde{h}\|^2_{L^2} \,. \tag{44}$$

Therefore, $\hat{h} = h$. Moreover, u satisfies (1c) so that $u \circ \eta_t = X_H(\eta_t)$, whence (43) implies that η_t is a flow of X_H. □

As is readily seen from Theorem 4.2 and Proposition 6, the existence of integral curves of the Hamiltonian vector field X_H on O implies the existence of solutions to the generalized LSG equations for nearly constant initial data. In particular, if one could show that X_H is a C^1 vector field, well-posedness of the equations would immediately follow from the theory of ODEs on Banach manifolds. Results of this type have been established, e.g., for the incompressible Euler equations [9] and for the Euler-α equations [23]. However, verifying the differentiability of the vector field X_H in our context appears to be difficult. These difficulties mostly vanish when studying well-posedness of the generalized LSG equations using more traditional techniques from PDE theory. This will be detailed in the final two sections of this paper.

5. **Hamiltonian formalism in Eulerian representation.** Despite the appearance of Eulerian quantities in the expressions for H and $\omega_{f,\lambda}$, the fundamental object in the Hamiltonian description developed in Section 4 is the flow map η rather than the Eulerian velocity u and layer depth h; therefore, this formalism is Lagrangian in nature. At the same time, the Euler equations of ideal fluid flow are Hamiltonian in the Eulerian as well as in the Lagrangian representation [2, 3, 9]. Hence, it is natural to expect that a similar duality holds for the generalized LSG equations.

Recall that the link between the Lagrangian and Eulerian Hamiltonian formalism for the Euler equation is established via Lie–Poisson reduction based on the particle relabeling symmetry [14, 25]. In contrast to the setting for the Euler equations, the phase space for the generalized LSG equations in the Lagrangian representation is not a tangent bundle of a Lie group, but the group itself. This necessitates the use of Poisson rather than Lie–Poisson reduction, while the symmetry remains the particle relabeling symmetry. This will be detailed in the following.

Consider the right action of the group of volume preserving diffeomorphisms on the full diffeomorphism group $\mathcal{D}^s$ given by $\Phi_\xi \eta = \eta \circ \xi$, where $\xi \in \mathcal{D}^s_{\text{vol}}$ and $\eta \in \mathcal{D}^s$. Since

$$J_{(\eta\circ\xi)^{-1}} = J_{\eta^{-1}} \,, \tag{45}$$

it is readily seen from the definitions that the symplectic form $\omega_{f,\lambda}$, its domain O, and the energy H are invariant with respect to this action, i.e.,

$$\Phi_\xi O = O \,, \qquad \Phi_\xi^* \omega_{f,\lambda} = \omega_{f,\lambda} \,, \qquad \text{and} \quad H(\Phi_\xi \eta) = H(\eta) \,. \tag{46}$$

The symplectic form $\omega_{f,\lambda}$ defines a Poisson bracket on O via

$$\{F, G\}_O = \omega_{f,\lambda}(X_F, X_G) \tag{47}$$

for smooth functions F, G on O. Let $\pi \colon O \to O/\mathcal{D}^s_{\text{vol}}$ be the projection onto the orbit space; in this context, it is called a reduction map. Invariances (46) imply that a Poisson bracket on the orbit space is given by

$$\{F, G\}(h) = \omega_{f,\lambda}(\pi^{-1}h)(X_{F\circ\pi}, X_{G\circ\pi}) \,. \tag{48}$$

The construction of the orbit space as a quotient manifold $\mathcal{D}^s/\mathcal{D}^s_{\text{vol}}$ is, in general, technically challenging (see [1] for the finite dimensional construction and [8, 13] for infinite-dimensional examples). Here, however, this difficulty disappears when one notices that π defined by $\pi(\eta) \equiv h(\eta) = J_{\eta^{-1}}$ is a readily available reduction map. Indeed, denote

$$H^{s-1}_+ = \{h \in H^{s-1}(D, \mathbb{R}) \mid h > 0\} \,, \tag{49a}$$

$$H^{s-1}_a = \{h \in H^{s-1}(D, \mathbb{R}) \mid \int_D h = a\} \,, \tag{49b}$$

$$H^{s-1}_J = H^{s-1}_+ \cap H^{s-1}_{\text{vol}(D)} \,. \tag{49c}$$

Then the map $\pi\colon \mathcal{D}^s \to H_J^{s-1}$ is surjective by Proposition 6 and, due to (45), it collapses each orbit $O_\eta = \{\Phi_\xi \eta \mid \xi \in \mathcal{D}^s_{\mathrm{vol}}\}$ into the single point $\pi(O_\eta) = J_{\eta^{-1}}$. To characterize the manifold structure of H_J^{s-1}, notice that H_+^{s-1} is an open set and $H^{s-1}_{\mathrm{vol}(D)}$ is an affine hyperplane in the Hilbert space $H^{s-1}(D,\mathbb{R})$. Therefore, H_J^{s-1} is diffeomorphic to an open set of the Hilbert space H_0^{s-1} and its tangent bundle is trivial, i.e., for any $h \in H_J^{s-1}$,

$$T_h H_J^{s-1} = H_0^{s-1} = \{z \in H^{s-1}(D,\mathbb{R}) \mid \int_D z = 0\}\,. \tag{50}$$

To complete the construction of the Poisson bracket by formula (48), we must clear two technical hurdles. First, we must differentiate π on the right of (48) which is merely continuous as a map from $\mathcal{D}^s$ to H_J^{s-1}. Note, however, that $\pi \in C^1(\mathcal{D}^s, H_J^{s-2})$. Indeed, differentiating π along the curve η_t with initial velocity $v \circ \eta \in T_\eta \mathcal{D}^s$ and using continuity equation (6), we obtain

$$T\pi \cdot (v \circ \eta) = \frac{\mathrm{d}}{\mathrm{d}t}_{t=0} J_{\eta_t^{-1}} = -\operatorname{div}(hv) \in H_J^{s-2}\,. \tag{51}$$

Second, $\omega_{f,\lambda}$ is merely a weak symplectic form. Hence, not every smooth function F may have an associated Hamiltonian vector field X_F. The solution is to restrict the bracket to a large enough subset of smooth functions that do have associated Hamiltonian vector fields and ensure that non-degeneracy of the bracket still holds on the chosen subset. We refer the reader to [25] for an example of such a construction.

We write $\tilde{O}$ to denote the image of the restriction of π to O, so that

$$\tilde{O} \equiv \pi(O) = \{h \in H_J^{s-1} \mid \|h-1\|_{s-1} < C\} \tag{52}$$

with C as in Proposition 5. Clearly, $\tilde{O}$ is an open subset of H_J^{s-1}. Define

$$\mathcal{F} = \{F \in C^\infty(\tilde{O},\mathbb{R}) \mid \mathrm{d}F(h) \in H^{s-1}(D,\mathbb{R}) \text{ for every } h \in \tilde{O}\}\,, \tag{53}$$

where we identify $H^{s-1}(D,\mathbb{R})$ with a subspace of $T_h^* H_J^{s-1}$ via the L^2 inner product. Let $F, G \in \mathcal{F}$, $w \circ \eta \in T_\eta \mathcal{D}^s$, and η_t be a curve through η with initial velocity $w \circ \eta$. Then, using equation (51),

$$\mathrm{d}(F\circ\pi)\cdot(w\circ\eta) = -\int_D \mathrm{d}F(h)\cdot \operatorname{div}(wh) = \int_D (\nabla \mathrm{d}F(h)\cdot w)\circ\eta\,. \tag{54}$$

Comparing (54) with (32), we find that the Hamiltonian vector field $X_{F\circ\pi}$ is given by

$$X_{F\circ\pi}(\eta) = \Lambda_h^{-1}\nabla^\perp \mathrm{d}F(\pi\eta)\,, \tag{55}$$

with Λ_h defined by (34). The invertibility of Λ_h will be discussed in Section 6. Here we remark that if $s > 3$, the constant C in the definition of $\tilde{O}$ can be chosen so that conditions of Proposition 8 hold; hence, the Hamiltonian vector field $X_{F\circ\pi}$ is well defined. The expression (48) for the reduced bracket becomes

$$\{F,G\}(h) = -\int_D \mathrm{d}F(h)\operatorname{div}(h\,\Lambda_h^{-1}\nabla^\perp \mathrm{d}G(h))\,. \tag{56}$$

The reduced Hamiltonian $\tilde{H}\colon \tilde{O} \to \mathbb{R}$ is given by the the formula

$$\tilde{H}(h) = H(\pi^{-1}h) = \frac{1}{2}\int_D \big(h^2 + 2\varepsilon\lambda\, h\,|\nabla h|^2\big)\,. \tag{57}$$

To proceed, we note that $\tilde{O}$ is open in H^{s-1}_J with trivial tangent bundle, i.e., $T\tilde{O} = \tilde{O} \times H^{s-1}_0$. Differentiating (57) in the direction $z \in H^{s-1}_0$, we obtain

$$\mathrm{d}\tilde{H}(h) \cdot z = \int_D \bigl(h - \varepsilon\lambda \,(2h\,\Delta h + |\nabla h|^2)\bigr)\, z\,. \tag{58}$$

The Hamiltonian vector field $X_{\tilde{H}}$ with respect to the bracket (56) satisfies

$$\{F, \tilde{H}\}(h) = \mathrm{d}F(h) \cdot X_{\tilde{H}}(h) = \int_D \mathrm{d}F(h)\, X_{\tilde{H}}(h) \tag{59}$$

for any $F \in \mathcal{F}$. Comparing (59) with (56) and substituting expression (58) for $\mathrm{d}\tilde{H}$, we obtain

$$X_{\tilde{H}}(h) = -\operatorname{div}\bigl[h\,\Lambda_h^{-1}\nabla^\perp\bigl(h - \varepsilon\lambda\,(2h\Delta h + |\nabla h|^2)\bigr)\bigr]\,. \tag{60}$$

It is now easy to verify that (60) implies the generalized LSG equations. Indeed, suppose that h_t is an integral curve of $X_{\tilde{H}}$. Setting $u = \Lambda_h^{-1}\nabla^\perp\bigl(h - \varepsilon\lambda(2h\Delta h + |\nabla h|^2)\bigr)$, which is precisely the u-h inversion equation (1c), we find that $\frac{\mathrm{d}}{\mathrm{d}t} h_t = X_{\tilde{H}}$ becomes the continuity equation (8b).

We remark that even when $\lambda = 0$, the Eulerian Hamiltonian vector field $X_{\tilde{H}}$ remains unbounded on H^{s-1}_J. On the other hand, its Lagrangian analogue $X_H(\eta) = (\Lambda_h^{-1}\nabla^\perp h) \circ \eta$ is at least continuous in the H^s topology. This behavior is expected due to non-differentiability of the reduction map and is typical for fluids. It is encountered in various other models of fluid motion such as, for example, incompressible and averaged Euler equations [9, 23].

6. **Kinematic estimates.** In the final two sections, we prove local well-posedness of the generalized LSG equations in their vorticity formulation (1). In this formulation, all problem-specific details are contained in the nonlinear kinematic relationship between the potential q and the velocity u expressed by (1c) and (1b). In this section, we state the necessary estimates on the problem of inverting these equations to obtain the velocity u in terms of a given potential vorticity q. The dynamic potential vorticity advection equation (1a) shall be discussed in Section 7.

For simplicity, we continue to assume a that D is a doubly-periodic domain. Here, moreover, we take $f = 1$ and $0 < \sigma \leq 1$. One can easily establish more general results with modified constants for non-flat manifolds and non-constant f provided the non-constant coefficient Helmholtz operator $f - \sigma\Delta$ remains an $H^{s+1} \to H^{s-1}$ isomorphism. This condition is satisfied, for example, if

$$\|f - 1\|_{s-1} < 1\,. \tag{61}$$

We will write $h \equiv \tilde{h} + 1$ and $q \equiv \tilde{q} + 1$ throughout. Finally, we recall that H^s is a topological algebra for $s > 1$, i.e., there is a family of constants C_s, which may be chosen non-increasing in s, such that

$$\|wv\|_s \leq C_s\,\|w\|_s\,\|v\|_s \tag{62}$$

for any $w, v \in H^s$.

Proposition 7. *Suppose $s > 1$, $\tilde{q} \in H^s(D)$, and $C_s\,\|\tilde{q}\|_s < 1$. Then there is a unique $h(\tilde{q}) \in H^{s+2}(D)$ satisfying (1b) with*

$$\|\tilde{h}\|_s \leq \frac{\|\tilde{q}\|_s}{1 - C_s\,\|\tilde{q}\|_s} \tag{63}$$

and

$$\|\tilde{h}\|_{s+2} \leq \frac{1}{\sigma} \frac{\|\tilde{q}\|_s}{1 - C_s \|\tilde{q}\|_s} . \tag{64}$$

Moreover, for any $0 < r < 1$, the map $\tilde{q} \mapsto h(\tilde{q})$ is uniformly continuous on

$$B^s_{(1-r)/C_s} \equiv \{\tilde{q} \in H^s(D) \mid C_s \|\tilde{q}\|_s < 1 - r\} \tag{65}$$

as a map from H^s into H^{s+2}.

Proof. Equation (1b) is equivalent to

$$\tilde{h} = (1 - \sigma\Delta)^{-1}(\tilde{q}\,\tilde{h} + \tilde{q}) \equiv F_q(\tilde{h}) . \tag{66}$$

For any $s \geq 0$, the inverse Helmholtz operator $(1 - \sigma\Delta)^{-1}$ has norm one as a map $H^s \to H^s$ and norm $1/\sigma$ as a map $H^s \to H^{s+2}$. We estimate

$$\|F_q(h_1) - F_q(h_2)\|_s \leq C_s \|\tilde{q}\|_s \|h_1 - h_2\|_s < (1 - r) \|h_1 - h_2\|_s \tag{67}$$

for some $r = r(\tilde{q}) \in (0, 1)$ and any $h_1, h_2 \in H^s(D)$. Hence, F_q is a contraction on $H^s(D)$, and, by the contraction mapping principle, has a unique fixed point $\tilde{h}$. A direct application of (66) shows that, in fact, $\tilde{h} \in H^{s+2}$.

Passing to the H^s norms on both sides of (66), we obtain

$$\|\tilde{h}\|_s \leq C_s \|\tilde{q}\|_s \|\tilde{h}\|_s + \|\tilde{q}\|_s \tag{68}$$

which is equivalent to (63). To establish (64), we take the H^{s+2} norm of (66) and use (63), estimating

$$\|\tilde{h}\|_{s+2} \leq \frac{1}{\sigma} \left(C_s \|\tilde{h}\|_s \|\tilde{q}\|_s + \|\tilde{q}\|_s \right) \leq \frac{\|\tilde{q}\|_s}{\sigma} \left(\frac{C_s \|\tilde{q}\|_s}{1 - C_s \|\tilde{q}\|_s} + 1 \right) . \tag{69}$$

In order to prove the uniform continuity of $\tilde{q} \mapsto h(\tilde{q})$, suppose that pairs (h_1, q_1) and (h_2, q_2) satisfy (1b) with $\tilde{q}_1, \tilde{q}_2 \in B^s_{(1-r)/C_s}$. Writing

$$\tilde{h}_1 - \tilde{h}_2 = (1 - \sigma\Delta)^{-1} \left(\tilde{q}_1 \, (\tilde{h}_1 - \tilde{h}_2) + (\tilde{h}_2 + 1) \, (\tilde{q}_1 - \tilde{q}_2) \right) , \tag{70}$$

passing to the H^s norms, and using (63), we obtain

$$\|\tilde{h}_1 - \tilde{h}_2\|_s \leq \frac{1 + C_s \|\tilde{h}_2\|_s}{1 - C_s \|\tilde{q}_1\|_s} \|\tilde{q}_1 - \tilde{q}_2\|_s \leq \frac{1}{r^2} \|\tilde{q}_1 - \tilde{q}_2\|_s . \tag{71}$$

This establishes the uniform continuity into H^s. Passing to the H^{s+2} norms in (70) and using (71), we obtain uniform continuity into H^{s+2}. □

Proposition 8. *Suppose $s > 2$ and $3C_{s-1} \|\tilde{h}\|_s < 1$. Then Λ_h defined by (34) is an isomorphism between $H^{s+1}(D)$ and $H^{s-1}(D)$ satisfying*

$$\|\Lambda_h^{-1} \psi\|_{s+1} \leq \frac{1}{\sigma} \frac{\|\psi\|_{s-1}}{1 - 3\, C_{s-1} \|\tilde{h}\|_s} \tag{72}$$

and

$$\|(\Lambda_{h_1}^{-1} - \Lambda_{h_2}^{-1}) \psi\|_{s+1} \leq \frac{1}{\sigma} \frac{3\, C_{s-1} \|\psi\|_{s-1}}{(1 - 3\, C_{s-1} \|\tilde{h}_1\|_s)(1 - 3\, C_{s-1} \|\tilde{h}_2\|_s)} \|\tilde{h}_1 - \tilde{h}_2\|_s \tag{73}$$

for every $h_1, h_2, h \in H^s$ and $\psi \in H^{s-1}$.

Proof. We use a fixed point argument as in the previous proposition. The equation $\Lambda_h u = \psi$ is equivalent to

$$u = (1 - \sigma\Delta)^{-1} \left(\sigma \, \tilde h \, \Delta u + 2\sigma \, \nabla \tilde h \cdot \nabla u + \psi\right) \equiv F_{h,\psi}(u) \, . \tag{74}$$

The map $F_{h,\psi}$ is an H^{s+1} contraction since, for $u_1, u_2 \in H^{s+1}$,

$$\begin{aligned} &\|F_{h,\psi}(u_1) - F_{h,\psi}(u_2)\|_{s+1} \\ &\qquad \le \frac{1}{\sigma} \left(\sigma \, C_{s-1} \, \|\tilde h\|_{s-1} \, \|u_1 - u_2\|_{s+1} + 2\sigma \, C_{s-1} \, \|\tilde h\|_s \, \|u_1 - u_2\|_s\right) \\ &\qquad \le 3 \, C_{s-1} \, \|\tilde h\|_s \, \|u_1 - u_2\|_{s+1} \, . \end{aligned} \tag{75}$$

This establishes the invertibility of Λ_h. Passing to the norms in (74), we obtain (72). Finally, suppose that $u_1 = \Lambda_{h_1}^{-1} \psi$ and $u_2 = \Lambda_{h_2}^{-1} \psi$. Then,

$$\begin{aligned} u_1 - u_2 &= \sigma \, (1 - \sigma\Delta)^{-1} \left(\tilde h_1 \, \Delta(u_1 - u_2) + 2 \, \nabla \tilde h_1 \cdot \nabla(u_1 - u_2)\right) \\ &\quad + \sigma \, (1 - \sigma\Delta)^{-1} \left((\tilde h_1 - \tilde h_2) \, \Delta u_2 + 2 \, \nabla(\tilde h_1 - \tilde h_2) \cdot \nabla u_2\right). \end{aligned} \tag{76}$$

Passing to the norms and estimating $\|u_2\|_{s+1}$ via (72), we obtain (73). □

Proposition 9. *Suppose $s > 2$, $\tilde h \in H^{s+2}(D)$, and $3 \, C_{s-1} \, \|\tilde h\|_s < 1$. Then equation (1c) has a unique solution $u \in H^{s+1}(D)$ satisfying*

$$\|u\|_{s+1} \le \frac{1}{1 - 3 \, C_{s-1} \, \|\tilde h\|_s} \left(\frac{\|\tilde h\|_s}{\sigma} + \frac{|\lambda|}{\lambda + 1/2} \, (2 + 3 \, C_s \, \|\tilde h\|_s) \, \|\tilde h\|_{s+2}\right) . \tag{77}$$

For any $0 < r < 1$ and $R > 0$, the map $\tilde h \mapsto u(\tilde h)$ is uniformly continuous on $B^s_{(1-r)/(3C_{s-1})} \cap B^{s+2}_R$ as a map from H^{s+2} into H^{s+1}.

If $\lambda = 0$, it is sufficient that $\tilde h \in H^s$. Then, $\tilde h \mapsto u(\tilde h)$ is uniformly continuous on $B^s_{(1-r)/(3C_{s-1})}$ as a map from H^s into H^{s+1}.

Proof. We rewrite (1c) as $u = \Lambda_h^{-1} \Psi(\tilde h)$ with

$$\Psi(\tilde h) = \nabla^\perp \left[\tilde h - \varepsilon \, \lambda \, (2 \, (1 + \tilde h) \, \Delta \tilde h + |\nabla \tilde h|^2)\right] . \tag{78}$$

A direct estimate, using $\|\nabla h\|_s^2 \le \|h\|_s \, \|h\|_{s+2}$, shows that

$$\|\Psi(\tilde h)\|_{s-1} \le \|\tilde h\|_s + \varepsilon \, |\lambda| \left(2 \, (1 + C_s \, \|\tilde h\|_s) \, \|\tilde h\|_{s+2} + C_s \, \|\tilde h\|_s \, \|\tilde h\|_{s+2}\right) . \tag{79}$$

By Proposition 8, this implies the existence of a unique solution u satisfying (77). Uniform continuity of $\tilde h \mapsto \Psi(\tilde h)$ is similarly checked, and implies uniform continuity of $\tilde h \mapsto u(\tilde h)$ as stated by Proposition 9. □

Combining propositions 7 and 9, we obtain a simple condition guaranteeing that both equations (1c) and (1b) can be solved simultaneously, thus defining u as a function of q.

Corollary 1. *Suppose $s > 2$, $\tilde q \in H^s(D)$, and $4 \, C_{s-1} \, \|\tilde q\|_s < 1$. Then there exist unique $h \in H^{s+2}(D)$ and $u \equiv K(q) \in H^{s+1}(D)$ satisfying (1c), (1b), and*

$$\|u\|_{s+1} \le \frac{1}{\sigma} \, \frac{\|\tilde q\|_s}{1 - 4 \, C_{s-1} \, \|\tilde q\|_s} \left(1 + \frac{3 \, |\lambda|}{\lambda + 1/2}\right) . \tag{80}$$

Moreover, for any $0 < r < 1$, the map $K \colon q \mapsto u$ is uniformly continuous on $B^s_{(1-r)/(4C_{s-1})}$ as a map from H^s into H^{s+1}.

Proof. Noting that C_s is decreasing in s, Proposition 7 asserts the existence of $h \in H^{s+2}$ satisfying (1b) with $\|\tilde{h}\|_s < (3C_{s-1})^{-1}$ and $\|\tilde{h}\|_{s+2} < (3\sigma C_{s-1})^{-1}$. Thus, Proposition 9 applies, asserting the existence and uniform continuity of K. Estimate (80) is obtained by substituting the appropriate bounds on $\tilde{h}$ into (77). □

7. **Local classical solutions.** In this section, we prove local-in-time existence of classical solutions to the full time-dependent problem (1).

Theorem 7.1. *Let $s > 2$ and let $m \in 1, \dots, \lfloor s-1 \rfloor$. Then for every initial datum $q(0) \in H^s(D)$ with $\|\tilde{q}(0)\|_s < (4C_{s-m})^{-1}$ there exists a unique local classical solution to the vorticity equations (1) of class*

$$q \in \bigcap_{k=0}^{m} C^k([0,T); H^{s-k}(D)) \,, \tag{81a}$$

$$h \in \bigcap_{k=0}^{m} C^k([0,T); H^{s+2-k}(D)) \,, \tag{81b}$$

and

$$u \in \bigcap_{k=0}^{m-1} C^k([0,T); H^{s+1-k}(D)) \,. \tag{81c}$$

If, in addition, $s > 3$ and $\|\tilde{q}(0)\|_{s-1} < (4C_{s-2})^{-1}$, the solution is unique.

We remark that such solutions also satisfy in a classical sense the Hamiltonian formulations $\frac{\mathrm{d}}{\mathrm{d}t} h_t = X_{\tilde{H}}$ of Section 5 and $\frac{\mathrm{d}}{\mathrm{d}t} \eta_t = X_H$ of Theorem 4.2. The condition on uniqueness is not sharp, but leads to a simple proof. For more refined estimates, see [5].

Proof. We only sketch the proof by demonstrating the necessary *a priori* estimates. To make this argument rigorous, one may construct the solution as the limit of a Galerkin approximating sequence. This is a lengthy but, as soon as the *a priori* estimates are obtained, routine procedure which has been implemented, for example, for the two-dimensional Euler equations in velocity formulation by Temam [24] and for a generalized vorticity-streamfunction formulation in [15]. We note that the formal *a priori* estimates stated below only refer to the boundedness of the nonlinear operator K. When passing to the limits of subsequences, however, uniform continuity as stated in all the kinematic estimates above is essential.

We write problem (1) in the form

$$\partial_t q + u \cdot \nabla q = 0 \,, \tag{82a}$$

$$u = K(q) \,, \tag{82b}$$

where the operator K is defined in Corollary 1. Further, we note the nonlinear estimates

$$\langle u \cdot \nabla q, q \rangle_s \le c_1 \, \|u\|_{s+1} \, \|q\|_s^2 \,, \tag{83a}$$

$$\langle u \cdot \nabla q, \psi \rangle_s \le c_2 \, \|u\|_{s+1} \, \|q\|_s \, \|\psi\|_{s+1} \,. \tag{83b}$$

for any $u, q, \psi \in H^{s+1}$. Now take the H^s inner product of (82a) with $\tilde{q}$ and use (80) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{q}\|_s^2 = -2 \, \langle u \cdot \nabla \tilde{q}, \tilde{q} \rangle_s \le 2 \, c_1 \, \|u\|_{s+1} \, \|\tilde{q}\|_s^2 \le c_3 \, \frac{\|\tilde{q}\|_s^3}{1 - 4 \, C_{s-1} \, \|\tilde{q}\|_s} \,. \tag{84}$$

Since the expression on the right is increasing in $\|\tilde{q}\|_s$, this differential inequality can be directly integrated. Noting that $C_{s-m} \geq C_{s-1}$, we find that there exists some $T > 0$ such that

$$\max_{t\in[0,T]} \|\tilde{q}(t)\|_s < \frac{1}{4\,C_{s-m}} \tag{85}$$

and, for every $t^* \in [0,T)$,

$$\limsup_{t\searrow t^*} \|\tilde{q}(t)\|_s \leq \|\tilde{q}(t^*)\|_s \,. \tag{86}$$

Finally, we take the H^s inner product of (82a) with a smooth test function ψ and integrate in time, so that

$$\begin{aligned} \langle \psi, \tilde{q}(t_2)\rangle_s - \langle \psi, \tilde{q}(t_1)\rangle_s &= \int_{t_1}^{t_2} \langle \psi, u\cdot\nabla\tilde{q}\rangle_s \,\mathrm{d}t \\ &\leq c_2 \int_{t_1}^{t_2} \|u\|_{s+1}\, \|\tilde{q}\|_s \,\|\psi\|_{s+1} \,\mathrm{d}t \\ &\leq c_4 \|\psi\|_{s+1} \int_{t_1}^{t_2} \frac{\|\tilde{q}\|_s^3}{1-4\,C_{s-m}\,\|\tilde{q}\|_s} \,\mathrm{d}t \,, \end{aligned} \tag{87}$$

where the first inequality is due to (83b) and the second due to (80). Noting the uniform bound (85), we see that the right hand side can be made arbitrarily small by making $|t_2 - t_1|$ sufficiently small. This proves weak continuity of $t \mapsto \tilde{q}(t)$ as a map into H^s.

Now we establish (81). Weak continuity and upper semi-continuity of the norm (86) imply continuity, i.e.,

$$q \in C([0,T]; H^s(D)) \,. \tag{88}$$

Then, Corollary 1 implies $u \in C([0,T]; H^s)$. Since H^{s-1} is a topological algebra, $q \mapsto u\cdot\nabla q$ is continuous as a map from H^s into H^{s-1} and (82a) implies that $\partial_t q \in C([0,T); H^{s-1})$. Moreover, Proposition 7 implies $h \in C([0,T); H^{s+2})$. Differentiating (1b) in time, we obtain

$$\partial_t h = (1-\sigma\Delta)^{-1}(1 - \tilde{q}\,\partial_t h - h\,\partial_t q) \,. \tag{89}$$

Hence, the argument in the proof of Proposition 7 applies, requiring only trivial modification of the contraction map F_q, which proves that $h \in C^1([0,T); H^{s+1})$. Similarly, differentiating (1c) in time, we obtain

$$\begin{aligned} \partial_t u = \Lambda_h^{-1} \nabla^\perp \partial_t \big[h - \varepsilon\,\lambda\,(2\,h\,\Delta h + |\nabla h|^2) \big] \\ + \sigma\,\Lambda_h^{-1} \big[\partial_t h\,\Delta u + 2\,\nabla\partial_t h \cdot \nabla u \big] \,. \end{aligned} \tag{90}$$

Thus, by Proposition 8, $u \in C^1([0,T); H^s)$ provided $s > 3$. This argument can be repeated where, to establish bounds on $\partial_t^k q$ in H^{s-k} and on $\partial_t^k h$ in H^{s-k+2}, we must require that H^{s-k} is a topological algebra; to establish a bound on $\partial_t^k u$ in H^{s-k+1}, we need the stronger restriction that H^{s-k-1} is a topological algebra. Altogether, this proves (81).

To prove uniqueness, consider two pairs (u_1, q_1) and (u_2, q_2) of solutions of (82) with the same initial condition $q_1(0) = q_2(0)$. Taking the H^{s-1} inner product of the equation for $q \equiv q_1 - q_2$, then using (83), (85), (80), and the uniform continuity of K, we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{s-1}^2 &= \langle u_2\cdot\nabla q, q\rangle_{s-1} + \langle (u_2-u_1)\cdot\nabla q_1, q\rangle_{s-1} \\ &\le c_4\,\|u_2\|_s\,\|q\|_{s-1}^2 + c_5\,\|u_2-u_1\|_s\,\|\tilde{q}_1\|_s\,\|q\|_{s-1} \\ &\le c_6\,\|q\|_{s-1}^2 + c_7\,\|q\|_{s-1}^2 \le c_8\,\|q\|_{s-1}^2\,. \end{aligned} \tag{91}$$

This estimate implies that $q \equiv 0$ on $[0,T]$ since it is initially so. □

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