STABILITY UNDER GALERKIN TRUNCATION OF A-STABLE RUNGE–KUTTA DISCRETIZATIONS IN TIME

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Abstract. We consider semilinear evolution equations for which the linear part is normal and generates a strongly continuous semigroup and the nonlinear part is sufficiently smooth on a scale of Hilbert spaces. We approximate their semiflow by an implicit, A-stable Runge–Kutta discretization in time and a spectral Galerkin truncation in space. We show regularity of the Galerkin-truncated semiflow and its time-discretization on open sets of initial values with bounds that are uniform in the spatial resolution and the initial value. We also prove convergence of the space-time discretization without any condition that couples the time step to the spatial resolution. Then we estimate the Galerkin truncation error for the semiflow of the evolution equation, its Runge–Kutta discretization, and their respective derivatives, showing how the order of the Galerkin truncation error depends on the smoothness of the initial data. Our results apply, in particular, to the semilinear wave equation and to the nonlinear Schrödinger equation.

Contents

1. Introduction

We study semilinear evolution equations

$$\partial_t U = F(U) = AU + B(U) \tag{1.1}$$

posed on a Hilbert space $\mathcal{Y}$ under spectral spatial Galerkin truncation and temporal discretization by a large class of A-stable Runge–Kutta methods. The methods

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considered permit a well-defined temporal semi-discretization [12, 15]; particular examples are Gauss–Legendre Runge–Kutta schemes. The linear operator A of (1.1) is assumed to be normal and to generate a strongly continuous, not necessary analytic semigroup; B is a bounded nonlinear operator on $\mathcal{Y}$. (This setting includes, without loss of generality, cases where A is normal up to a bounded perturbation as a bounded non-normal part can always be included into the operator B). The examples we have in mind are semilinear Hamiltonian evolution equations such as the semilinear wave equation or the nonlinear Schrödinger equation, though for the results in this paper we do not assume a Hamiltonian structure.

Differentiation of the semiflow in time results in multiplication with the unbounded operator A. Hence, in general, the time derivative of the semiflow is only well-defined when considered as a map from a subset of $D(A)$ to $\mathcal{Y}$ [16]; to be able to differentiate repeatedly, we assume that B is $\mathcal{C}^{N-k}$ as map from some open set $\mathcal{D}_k \subset \mathcal{Y}_k \equiv D(A^k)$ to $\mathcal{Y}_k$ for $k = 0, \ldots, K$ and $N > K$. This is formalized as condition (B1) in the main text of the paper. We prove that the semiflow of the Galerkin truncated evolution equation and its temporal discretization are of class $\mathcal{C}^K$ jointly in time (resp. stepsize) and in the initial data when considered as a map from $\mathcal{D}_K \subset \mathcal{Y}_K$ to $\mathcal{Y}$, with uniform bounds in the spatial resolution. Analogous results hold true for the full semiflow and its time-semidiscretization [15]. We prove full-order convergence of the space-time discretization on open sets of initial data without the need of a Courant condition that couples spatial and temporal resolution. We then provide estimates on the truncation error of the Galerkin approximation of the semiflow, the temporal discretization and their derivatives, and study the dependence of the order of the truncation error on the smoothness of the initial data.

When implicit Runge–Kutta methods are applied to stiff problems, they often converge at less than their formal order of convergence. This phenomenon is called *order reduction* [6]. For time-semidiscretizations of initial-boundary value problems, order-reduction can be tied to lack of regularity [5] or mismatch of boundary conditions in the internal stages of the method [2]; both papers give conditions under which full-order convergence is achieved. In our work, we are in the setting of [5, Theorem 3] except that we consider semilinear equations. For linear evolution equations, this earlier result gives order p convergence when p is the formal order of the method and the initial data is in $D(A^{p+1})$. When considering semilinear problems, our condition (B1) on the mapping properties of the nonlinearity typically imposes additional boundary conditions that the nonlinearity B has to match if the operator A has boundary conditions which are not periodic. Condition (B1), together with the assumption that the initial data lie in $D(A^{p+1})$, enforces matching boundary conditions and excludes order reduction.

The standard requirement for the existence of a semiflow is Lipshitz continuity of the nonlinearity B [16]. It holds true for a large class of evolution equations and will be referred to as condition (B0) in the main text of the paper. Our assumption (B1) implies Lipshitz continuity. Whether the stronger condition (B1) holds true with $K > 0$ depends nontrivially on the evolution equation and its boundary conditions. It is satisfied by our main examples, the semilinear wave equation and nonlinear Schrödinger equation with smooth nonlinearities and periodic or homogeneous Neumann boundary conditions. It is also satisfied for homogeneous Dirichlet conditions under additional conditions on the nonlinearity, see Section 2 below. If

(B1) is not satisfied for sufficiently large K, we cannot ensure that the solution $U(t)$ of (1.1) and its numerical approximations have enough temporal smoothness to obtain full order convergence of the time-discretization independent of the spatial resolution.

We recall from [16] that the solution to the full semilinear evolution equation is obtained as a fixed point of a contraction map, which we consider on the scale of Hilbert spaces $\mathcal{Y}_0, \dots, \mathcal{Y}_K$. Similarly, the Runge–Kutta temporal discretizations are functions of the Runge–Kutta stage vectors, which in turn are obtained as fixed points of contraction maps. Remaining in this setting, we now consider spatial Galerkin approximation as a perturbation of these contraction maps. To do so, we provide an abstract theory for the stability of fixed points under perturbation of contraction mappings on a scale of Banach spaces, thereby extending the theory of contraction maps on scales of Banach spaces from [15, 19, 21]. This theory provides us with a unified framework for the time-continuous and the time-discrete case.

Let us mention some related results. Spatial spectral Galerkin approximation (also called Faedo–Galerkin approximation) is frequently used as a theoretical tool for the construction of solutions to partial differential equations; see, e.g., [9, 17]. Error estimates for smooth solutions of parabolic problems under spectral and more general Galerkin approximations (such as finite element methods) can be found, e.g., in [10, 18]. In the parabolic case, there has been a lot of interest in the so-called nonlinear Galerkin method which has been shown to have a better convergence rate than the standard spectral Galerkin method, see [7, 13] and references therein. For analytic initial data, an exponential rate of convergence of the Galerkin approximation to the semiflow of the Ginzburg–Landau equation has been shown in [8].

Hyperbolic problems, namely the semilinear wave equation, and their discretizations have been studied, e.g., by Baker *et al.* [3]. They provide estimates for the order of convergence of the spatial Galerkin method of the semiflow for smooth enough data and globally Lipshitz nonlinearities under an assumption on the elliptic projection of the solution; they also treat explicit multistep time discretizations of the spatial approximation under a Courant condition that couples the accuracy of the Galerkin method with the time-stepsize. Bazley [4] shows the convergence of the Faedo–Galerkin approximations of the semilinear wave equation for a special class of nonlinearities on the interval of existence of the continuous solution. Verver and Sanz-Serna [20] identify general consistency and stability conditions in which convergence of spatial semidiscretizations and of their temporal discretizations can be proved. They further verify these conditions for a nonlinear parabolic PDE and for the cubic nonlinear Schrödinger equation. In this paper we provide a general framework in which those conditions hold true with uniform bounds on open sets of initial data. Miklavcic [14] studies a class of parabolic and hyperbolic semilinear evolution equations with a linear part that generates a $\mathcal{C}^0$ semigroup, and shows pointwise convergence of the spatial Galerkin approximations of the semiflow; he considers nonlinearities B which are Lipshitz on the whole of $\mathcal{Y}$. Karakashian *et al.* [11] study a class of implicit Runge–Kutta time-discretizations (including Gauss–Legendre methods) and spatial Galerkin approximations for the cubic nonlinear Schrödinger equations and prove convergence for smooth solutions under mesh conditions that couple spatial and temporal resolution.

In this paper, the emphasis is on estimates for the spatial Galerkin truncation error of the joint higher order derivatives in time and in the initial data both of the semiflow and of its temporal discretization. In contrast to [3, 11], our estimates for the numerical method hold uniformly in the time-stepsize and do not require conditions that couple the spatial and temporal accuracy of the discretization. Our results include statements on the pointwise convergence of Galerkin spatial semi-discretizations for non-smooth solutions of (1.1) on their interval of existence, see Theorem 2.3. These are similar to the results of [4, 14], but include more general evolution equations. Our results yield algebraic orders of the Galerkin truncation error for smooth, but non-analytic initial data. However, using the methods developed here, it is also possible to obtain exponential estimates for analytic data as in [8].

The paper is organized as follows. In Section 2, we introduce the class of semilinear evolution equations considered, and show how the semilinear wave equation and the nonlinear Schrödinger equation fit into this framework for different types of boundary conditions. In this setting, we study regularity and stability under Galerkin truncation of the semiflow. In Section 3, we apply a class of A-stable Runge–Kutta methods to the semiflow of the Galerkin truncated evolution equation and prove results on regularity and stability under Galerkin truncation for the temporal discretization which are analogous to the semiflow. We also study convergence of the space-time discretization.

We present our results in two versions: we label results that provide uniformity of the time interval of existence (for the semiflow) and the maximum time step (for the numerical method) on sufficiently small balls of initial data as "local version." Assuming more regularity for the initial data, we also obtain results which are uniform on bounded open sets so long as B is well-defined and bounded. We will label results of this type by "uniform version."

In the appendix, we present a number of technical results on stability of fixed points of contraction maps on scales of Banach spaces which are needed in the main body of the paper.

2. Semilinear evolution equations under Galerkin truncation

We begin by introducing the class of semilinear evolution equations which we study in this paper. We then prove regularity of the Galerkin truncated semiflow with uniform bounds in its spatial resolution and analyze the dependency of the truncation error of the semiflow and its derivatives on the smoothness of the initial data.

2.1. **General setting.** We consider the semilinear evolution equation (1.1) on a Hilbert space $\mathcal{Y}$ and assume the following.

(A) A is a normal operator on a Hilbert space $\mathcal{Y}$ which generates a $\mathcal{C}^0$-semigroup e^{tA}.

(B0) $B\colon \mathcal{D} \to \mathcal{Y}$ is Lipshitz.

Recall that an operator A is normal if it is closed and $AA^* = A^*A$. For a definition of strongly continuous semigroups ($\mathcal{C}^0$-semigroups), see [16]. Assumption (A) implies that there exists $\omega \in \mathbb{R}$ such that

$$\operatorname{Re}(\operatorname{spec} A) \leq \omega \qquad \text{and} \qquad \|\mathrm{e}^{tA}\| \leq \mathrm{e}^{\omega t} \tag{2.1}$$

for all $t \geq 0$. In case $A = A_\mathrm{n} + A_\mathrm{b}$ where A_n satisfies (A) and A_b is bounded, we can redefine B as $B + A_\mathrm{b}$ and and A as A_n, whence conditions (A) and (B0) hold true. This situation is typical for semilinear wave equations, see Example 2.1.

For fixed $T > 0$ and $U^0 \in \mathcal{D}$ let $W \in \mathcal{C}([0,1]; \mathcal{Y})$ satisfy the fixed point equation $W = \Pi(W; U^0, T)$ where, for $\tau \in [0,1]$,

$$\Pi(W; U^0, T)(\tau) = \mathrm{e}^{\tau T A} U^0 + T \int_0^\tau \mathrm{e}^{(\tau - \sigma) T A} B(W(\sigma)) \, \mathrm{d}\sigma \,. \tag{2.2}$$

When T is small enough, Π is a contraction on the space $\mathcal{C}_\mathrm{b}([0,1]; \mathcal{Y})$ so that the contraction mapping theorem implies the existence of a fixed point [16]. We then define the semiflow Φ of (1.1) by $\Phi^{\tau T}(U^0) = W(U^0, T)(\tau)$. We sometimes write Φ^t to denote the map $\Phi(\,\cdot\,, t)$.

It is apparent from (2.2) with $B = 0$ that the ℓ-th time derivative of $U(t)$ is in $\mathcal{Y}$ only if $U^0 \in D(A^\ell)$. Hence, we work on a hierarchy of Hilbert spaces defined as follows. We set $\mathcal{Y}_0 \equiv \mathcal{Y}$ and, for $\ell \in \mathbb{N}$, we define $\mathcal{Y}_\ell \equiv D(A^\ell)$ endowed with scalar product

$$\langle U_1, U_2 \rangle_{\mathcal{Y}_\ell} = \langle \mathbb{P} U_1, \mathbb{P} U_2 \rangle_{\mathcal{Y}} + \langle |A|^\ell \, \mathbb{Q} U_1, |A|^\ell \, \mathbb{Q} U_2 \rangle_{\mathcal{Y}} \,.$$

Here $\mathbb{P} \equiv \mathbb{P}_1$ is the spectral projector of A onto the set $\{\lambda \in \operatorname{spec}(A) \colon |\lambda| \leq 1\}$, and $\mathbb{Q} = 1 - \mathbb{P}$. This definition of the norm ensures that

$$\|A\|_{\mathcal{Y}_{\ell+1} \to \mathcal{Y}_\ell} \leq 1 \quad \text{and} \quad \|U\|_{\mathcal{Y}_\ell} \leq \|U\|_{\mathcal{Y}_{\ell+1}} \tag{2.3}$$

for all $U \in \mathcal{Y}_{\ell+1}$.

Let $\mathcal{D} \subset \mathcal{Y}$ be open. We define

$$\mathcal{D}^{-\delta} = \{U \in \mathcal{D} \colon \operatorname{dist}_{\mathcal{Y}}(U, \partial \mathcal{D}) > \delta\} \,. \tag{2.4}$$

Given $\delta > 0$ and a hierarchy of open sets $\mathcal{D}_\ell \subset \mathcal{Y}_\ell$ for $\ell = 0, \ldots, L$ for $L \in \mathbb{N}$ with $\mathcal{D}_0 \equiv \mathcal{D}$, we define $\mathcal{D}_0^{-\delta} \equiv \mathcal{D}^{-\delta}$ as in (2.4) and, for $\ell = 1, \ldots, L$,

$$\mathcal{D}_\ell^{-\delta} \equiv \{U \in \mathcal{D}_\ell \colon \operatorname{dist}_{\mathcal{Y}_\ell}(U, \partial \mathcal{D}_\ell) > \delta\} \,. \tag{2.5}$$

Then, by construction, $\mathcal{B}_\delta^{\mathcal{Y}_\ell}(U) \subset \mathcal{D}_\ell$ for all $U \in \mathcal{D}_\ell^{-\delta}$ and $\ell = 0, \ldots, L$ where, for any Banach space $\mathcal{X}$ and $X^0 \in \mathcal{X}$, we write

$$\mathcal{B}_R^{\mathcal{X}}(X^0) = \{X \in \mathcal{X} \colon \|X - X^0\|_{\mathcal{X}} \leq R\}$$

to denote the closed ball of radius R around X^0.

Let $\mathcal{Y}_1$ be a Banach space continuously embedded into the Banach space $\mathcal{Y}$. Then $\mathcal{D}_1 \subset \mathcal{Y}_1$ is called a δ_*-nested subset of $\mathcal{D} \subset \mathcal{Y}$ if $\mathcal{D}_1^{-\delta} \subset \mathcal{D}^{-\delta}$ for all $\delta \in [0, \delta_*]$. Furthermore we say that the family $\mathcal{D}_0, \ldots, \mathcal{D}_L$ is δ_*-nested if $\mathcal{D}_\ell^{-\delta} \subset \mathcal{D}_{\ell-1}^{-\delta}$ for all $\delta \in [0, \delta_*]$ with $\delta_* > 0$ and $\ell = 1, \ldots, L$. For example, the family $\mathcal{D}_k = \mathcal{B}_R^{\mathcal{Y}_k}(U^0)$ is δ_*-nested for every $\delta_* \in (0, R)$ and $U^0 \in \mathcal{Y}_L$. However, an arbitrary nested family $\mathcal{D}_\ell \subset \mathcal{Y}_\ell$ may not be δ_*-nested for any $\delta_* > 0$.

Finally, we write $B \in \mathcal{C}_\mathrm{b}^N(\mathcal{D}, \mathcal{Y})$ for some $\mathcal{D} \subset \mathcal{Y}$ if $B \in \mathcal{C}^N(\mathcal{D}, \mathcal{Y})$ and if its derivatives are, in addition, bounded and extend continuously to the boundary. Then we can state a condition under which Φ defines a semiflow on the scale of spaces $\mathcal{Y}_0, \ldots, \mathcal{Y}_K$.

(B1) There exist $K \in \mathbb{N}_0$, $N \in \mathbb{N}$ with $N > K$, and a δ_*-nested sequence of $\mathcal{Y}_k$-bounded and open sets $\mathcal{D}_k \subset \mathcal{Y}_k$ such that $B \in \mathcal{C}_\mathrm{b}^{N-k}(\mathcal{D}_k, \mathcal{Y}_k)$ for $k = 0, \ldots, K$.

We denote the bounds of the maps $B\colon \mathcal{D}_k \to \mathcal{Y}_k$ and their derivatives by constants M_k, M_k', etc., for $k = 0, \dots, K$ and set $M = M_0$, $M' = M_0'$, and so forth. In addition to the domains $\mathcal{D}_0, \dots, \mathcal{D}_K$ defined in this assumption, we will sometimes need to refer to $\mathcal{D}_{K+1}$, which may be any δ_*-nested subset of $\mathcal{D}_K$ which is bounded and open in $\mathcal{Y}_{K+1}$.

We now give two examples of PDEs that satisfy assumptions (A) and (B1).

Example 2.1 (Functional setting for the semilinear wave equation). For the semilinear wave equation

$$\partial_{tt} u = \partial_{xx} u - f(u) \tag{2.6}$$

on $I = (0,1)$ with periodic boundary conditions $u(0) = u(1)$, we set $U = (u, v)$ and

$$\mathcal{Y}_\ell = \mathcal{H}_{\ell+1}(I; \mathbb{R}) \times \mathcal{H}_\ell(I; \mathbb{R})$$

for $\ell \in \mathbb{N}$. Here, $\mathcal{H}_\ell(I; \mathbb{R})$ denotes the Sobolev space of square integrable functions whose first ℓ weak derivatives are square-integrable. Then the operators A and B are given by

$$\tilde{A} = \begin{pmatrix} 0 & \mathrm{id} \\ \partial_x^2 & 0 \end{pmatrix}, \quad A = (1 - \mathbb{P}_0)\tilde{A}, \quad \text{and} \quad B(U) = \begin{pmatrix} u \\ -f(u) \end{pmatrix}, \tag{2.7}$$

where $\mathbb{P}_0$ is the spectral projector of $\tilde{A}$ to the eigenvalue 0. Note that we have moved $\mathbb{P}_0 \tilde{A} U$ into the nonlinearity B as $\mathbb{P}_0 \tilde{A}$ is not normal. Then the group generated by A is unitary on any $\mathcal{Y}_\ell$ and A generates a $\mathcal{C}^0$-group on $\mathcal{Y}_\ell$. So, assumption (A) is satisfied. If the nonlinearity f of the semilinear wave equation (2.6) is, e.g., a polynomial, then (B1) is satisfied for any K and N as $\mathcal{H}_\ell$ is a topological algebra for $\ell > 1/2$ [1]. More generally, if $f \in \mathcal{C}^N(D, \mathbb{R})$ for some $N \in \mathbb{N}$ and $D \subset \mathbb{R}$ open, then (B1) holds for $K < N$; see, e.g., [15, Theorem 2.12]. The same holds true in the case of homogeneous Neumann boundary conditions. For homogeneous Dirichlet conditions we must additionally require that $f^{(2j)}(0) = 0$ for $0 \le 2j \le K - 1$; the same restriction on the nonlinearity applies when A is a nonconstant coefficient operator and $K \le 4$, see [15, Section 2.5].

Example 2.2 (Functional setting for the nonlinear Schrödinger equation). For the nonlinear Schrödinger equation

$$\mathrm{i}\, \partial_t u = -\partial_{xx} u + \partial_{\overline{u}} V(u, \overline{u}) \tag{2.8}$$

with periodic boundary conditions on $I = (0,1)$, we set $U \equiv u$ and identify

$$A = \mathrm{i}\, \partial_x^2 \quad \text{and} \quad B(U) = -\mathrm{i}\, \partial_{\overline{u}} V(u, \overline{u}) \,. \tag{2.9}$$

The Laplacian is diagonal in the Fourier representation with eigenvalues $-k^2$ where $k \in \mathbb{Z}$. Hence A generates a unitary group on the square integrable functions $\mathcal{L}_2 \equiv \mathcal{L}_2(I; \mathbb{C})$ and, more generally, on every $\mathcal{H}_\ell(I; \mathbb{C})$ with $\ell \in \mathbb{N}_0$. So the operator A is normal, and assumption (A) holds trivially. In the notation of the abstract functional setting of Section 2.1, we choose $\mathcal{Y}_\ell = \mathcal{H}_{2\ell+1}(I; \mathbb{C})$. If the potential $V(u, \overline{u})$ satisfies $V \in \mathcal{C}^{K+2+N}(D, \mathbb{R}^2)$ for some open subset $D \subset \mathbb{R}^2 \equiv \mathbb{C}$, then, by [15, Theorem 2.12], the nonlinearity B defined in (2.9) satisfies assumption (B1) for $K < N$ and, in particular, (B0).

2.2. **Spectral Galerkin truncation and convergence.** We now truncate the evolution equation (1.1) to an A-invariant subspace (Galerkin subspace) as follows. For $m \in \mathbb{N}$ let $\mathbb{P}_m$ be the sequence of spectral projectors of A onto the set $\{\lambda \in \operatorname{spec}(A) \colon |\lambda| \leq m\}$. Then, assumption (A) implies that

$$\lim_{m\to\infty} \mathbb{P}_m U = U$$

for all $U \in \mathcal{Y}$, and

$$\|A\mathbb{P}_m U\|_{\mathcal{Y}} \leq m \, \|\mathbb{P}_m U\|_{\mathcal{Y}} \tag{2.10}$$

for $m \in \mathbb{N}$. Functions $U \in \mathcal{Y}_\ell$ are well approximated by their Galerkin projections $\mathbb{P}_m U$. Indeed, setting $\mathbb{Q}_m = \operatorname{id} - \mathbb{P}_m$,

$$\|\mathbb{Q}_m U\|_{\mathcal{Y}} \leq m^{-\ell} \, \|U\|_{\mathcal{Y}_\ell} \, . \tag{2.11}$$

We now introduce the restricted evolution equation

$$\begin{aligned} \dot{u}_m &= A u_m + B_m(u_m) = \mathbb{P}_m F(u_m) \\ &\equiv f_m(u_m) = A u_m + B_m(u_m) \, , \end{aligned} \tag{2.12}$$

where $B_m = \mathbb{P}_m B$. We write $\phi_m^t(\cdot)$ to denote the semiflow of (2.12) on $\mathbb{P}_m \mathcal{Y}$ and define $\Phi_m = \phi_m \circ \mathbb{P}_m$.

The following theorem provides well-posedness for the projected system on the same interval of time on which a solution to the full equation exists, and convergence of solutions.

Theorem 2.3 (Convergence of the projected system)**.** *Under assumptions* (A) *and* (B0)*, let $U \in \mathcal{C}([0,T],\mathcal{D})$ be a mild solution to the semilinear evolution equation* (1.1) *with initial value $U(0) = U^0$. Then there is $m_* \in \mathbb{N}$ such that for every $m \geq m_*$ there exists a solution $u_m \in \mathcal{C}([0,T],\mathcal{D})$ to the projected system* (2.12) *with initial value $u_m(0) = \mathbb{P}_m U^0$. Moreover,*

$$\sup_{t\in[0,T]} \|U(t) - u_m(t)\|_{\mathcal{Y}} \to 0 \tag{2.13}$$

as $m \to \infty$.

Proof. Local existence of a solution $u_m(t)$ of (2.12) is obvious since A_m is bounded. However, we need to show that the interval of existence is at least $[0,T]$. We note that the solution can only cease to exist if u_m leaves the domain $\mathcal{D}$, so we proceed to prove (2.13) directly. Clearly,

$$U(t) - u_m(t) = \mathrm{e}^{tA} \mathbb{Q}_m U^0 + \int_0^t \mathrm{e}^{(t-s)A} \left(B(U(s)) - B_m(u_m(s)) \right) \mathrm{d}s \, . \tag{2.14}$$

Taking the $\mathcal{Y}$-norm and noting that, by (2.1), there is $c > 0$ such that $\|\mathrm{e}^{tA}\|_{\mathcal{E}(\mathcal{Y})} \leq c$ for $t \in [0,T]$, we find that

$$\begin{aligned} \|U(t) - u_m(t)\|_{\mathcal{Y}} &\leq c \, \|\mathbb{Q}_m U^0\|_{\mathcal{Y}} + c \int_0^t \|B(U(s)) - B_m(u_m(s))\|_{\mathcal{Y}} \, \mathrm{d}s \\ &\leq c \, \|\mathbb{Q}_m U^0\|_{\mathcal{Y}} + c \, T \sup_{s\in[0,T]} \|\mathbb{Q}_m B(U(s))\|_{\mathcal{Y}} + c \int_0^t \|B(U(s)) - B(u_m(s))\|_{\mathcal{Y}} \, \mathrm{d}s \, . \\ &\leq c \, \|\mathbb{Q}_m U^0\|_{\mathcal{Y}} + c \, T \sup_{s\in[0,T]} \|\mathbb{Q}_m B(U(s))\|_{\mathcal{Y}} + c \, M_0' \int_0^t \|U(s) - u_m(s)\|_{\mathcal{Y}} \, \mathrm{d}s \, . \end{aligned} \tag{2.15}$$

Now note that the sequence of functions $f_m(s) = \|\mathbb{Q}_m B(U(s))\|_{\mathcal{Y}}$ converges pointwise to zero as $m \to \infty$. Moreover, since

$$|f_m(s_1) - f_m(s_2)| \leq \|\mathbb{Q}_m(B(U(s_1)) - B(U(s_2)))\|_{\mathcal{Y}} \leq \|B(U(s_1)) - B(U(s_2))\|_{\mathcal{Y}} ,$$

the sequence is uniformly equicontinuous. Hence, by the Arzelà–Ascoli theorem, f_m converges to zero uniformly as $m \to \infty$. Thus, applying the Gronwall inequality to (2.15), we see for any $\varepsilon > 0$ there exists a possibly larger m_* such that for $m \geq m_*$, $\|U(t) - u_m(t)\|_{\mathcal{Y}} \leq \varepsilon$ so long as $u_m(t)$ does not leave $\mathcal{D}$. Choosing $\varepsilon < \operatorname{dist}(\{U(s)\colon s \in [0,T]\}, \partial\mathcal{D})$, we conclude that t in this estimate may be chosen as large as T. □

We now define

$$R_{K+1} = \sup_{U \in \mathcal{D}_{K+1}} \|U\|_{\mathcal{Y}_{K+1}} . \tag{2.16}$$

The following theorem provides higher order bounds for the Galerkin approximation error of the semiflow.

Corollary 2.4 (Convergence of the projected system – higher order error bounds)**.** *Assume* (A) *and* (B1)*. Let $\delta \in (0, \delta_*]$ be such that $\mathcal{D}^{-\delta}_{K+1}$ is nonempty. Then there exists m_* such that for all $m \geq m_*$ and every mild solution $U \in \mathcal{C}([0,T]; \mathcal{D}^{-\delta}_{K+1})$ of the semilinear evolution equation* (1.1) *there exists a solution $u_m \in \mathcal{C}([0,T], \mathcal{D}_K \cap \mathcal{Y}_{K+1})$ to the projected system* (2.12) *with initial value $u_m(0) = \mathbb{P}_m U^0$ such that*

$$\sup_{t \in [0,T]} \|U(t) - u_m(t)\|_{\mathcal{Y}} = O(m^{-K-1}) \tag{2.17}$$

The order constants in (2.17) *depend only on the bounds afforded by* (B1)*,* (2.1)*, and* (2.16)*, on δ, and on T.*

Proof. As in the proof of Theorem 2.3, we begin with (2.14). Here, we apply $\mathbb{P}_m$ and rearrange terms to obtain the estimate

$$\|U(t) - u_m(t)\|_{\mathcal{Y}} \leq \|\mathbb{Q}_m U(t)\|_{\mathcal{Y}} + c \int_0^t \|B(U(s)) - B(u_m(s))\|_{\mathcal{Y}} \, \mathrm{d}s . \tag{2.18}$$

Due to (2.11), $\|\mathbb{Q}_m U(\cdot)\|_{\mathcal{Y}} \leq R_{K+1} m^{-K-1}$. The mean value theorem applies so long as $u_m(s) \in \mathcal{D}$. Then, by the Gronwall lemma as before, we find that (2.17) holds true for all $m \geq m_*$, where we choose m_* such that $\|U(t) - u_m(t)\|_{\mathcal{Y}} < \delta$ for $t \in [0,T]$ and $m \geq m_*$ so that indeed $u_m(s) \in \mathcal{D}$ for $s \in [0,T]$ and $m \geq m_*$. □

Note that Corollary 2.4 with $\mathcal{Y}$ replaced by any of the $\mathcal{Y}_1, \dots, \mathcal{Y}_K$ readily implies that $\sup_{t \in [0,T]} \|U(t) - u_m(t)\|_{\mathcal{Y}_j} = O(m^{-K-1-j})$. However, as B is not assumed to map from an open subset of $\mathcal{Y}_{K+1}$ to $\mathcal{Y}_{K+1}$, Theorem 2.3 as stated does not apply with $\mathcal{Y}$ replaced by $\mathcal{Y}_{K+1}$. However, we can still prove the following.

Corollary 2.5. *Under the assumptions of Corollary 2.4, the following is true.*

(a) *If $N > K+1$, we have $\sup_{t \in [0,T]} \|U(t) - u_m(t)\|_{\mathcal{Y}_{K+1}} \to 0$ as $m \to \infty$.*
(b) *If $N = K+1$, there exists $C > 0$ such that $\sup_{t \in [0,T]} \|u_m(t)\|_{\mathcal{Y}_{K+1}} \leq C$. The bound C depends only on the bounds afforded by* (B1)*,* (2.1)*, and* (2.16)*, on δ, and on T.*

Proof. We may assume without loss of generality, that $K = 0$. (Otherwise replace $\mathcal{Y}$ with $\mathcal{Y}_K$.) Suppose first that $N > K + 1$. Then Theorem 2.3 applies to the system of evolution equations

$$\dot{U} = AU + B(U)\,, \qquad \dot{W} = AW + B'(U)W$$

with initial value $W(0) = AU^0 + B(U^0)$ so that $W(t) = U'(t)$. Hence,

$$\sup_{t\in[0,T]} \|U'(t) - u_m'(t)\|_{\mathcal{Y}} \to 0$$

as $m \to \infty$. Since $\sup_{t\in[0,T]}\|B(U(t)) - \mathbb{P}_m B(u_m(t))\|_{\mathcal{Y}} = O(m^{-1})$ by Corollary 2.4 and $AU(t) = U'(t) - B(U(t))$, we obtain statement (a).

To prove statement (b), integrate $\dot{w}_m = Aw_m + B'(u_m)w_m$ with initial value $w_m(0) = A\mathbb{P}_m U^0 + \mathbb{P}_m B(u_m(0))$ and apply a standard Gronwall argument as before, noting that the $\mathcal{Y}$-norm of $u_m(t)$ is bounded uniformly in $m \geq m_*$ by Corollary 2.4. Thus, $\sup_{t\in[0,T]}\|u_m'(t)\|_{\mathcal{Y}} \leq c$ for some $c > 0$ depending only on the bounds afforded by (B1), (2.1), and (2.16), on δ, and on T. This, together with the bound of B on $\mathcal{D}$, proves (b). □

2.3. **Regularity of Galerkin truncated semiflow.** We first introduce some notation. For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, and $j \in \mathbb{N}_0$, we write $\mathcal{E}^j(\mathcal{Y},\mathcal{X})$ to denote the vector space of j-multilinear bounded mappings from $\mathcal{Y}$ to $\mathcal{X}$; we set $\mathcal{E}^j(\mathcal{X}) \equiv \mathcal{E}^j(\mathcal{X},\mathcal{X})$. For Banach spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, and subsets $\mathcal{U} \subset \mathcal{X}$, $\mathcal{V} \subset \mathcal{Y}$, and $\mathcal{W} \subset \mathcal{Z}$, we write

$$F \in \mathcal{C}_{\mathrm{b}}^{(\underline{m},n)}(\mathcal{U} \times \mathcal{V}; \mathcal{W})$$

to denote a continuous, bounded function $F\colon \mathcal{U} \times \mathcal{V} \to \mathcal{W}$ whose partial Fréchet derivatives $\mathrm{D}_X^i \mathrm{D}_Y^j F(X,Y)$ exist, are bounded, and are such that the maps

$$(X, Y, X_1, \ldots, X_i) \mapsto \mathrm{D}_X^i \mathrm{D}_Y^j F(X,Y)(X_1,\ldots,X_i) \tag{2.19}$$

are continuous from $\mathcal{U} \times \mathcal{V} \times \mathcal{X}^i$ into $\mathcal{E}^j(\mathcal{Y},\mathcal{Z})$ for $i = 0,\ldots,m$ and $j = 0,\ldots,n$ and provided the maps (2.19) extend continuously to the boundary. (The latter is important as we will apply the contraction mapping theorem to maps in such classes.) In our setting, $\mathcal{V}$ will typically be an interval of time.

The following theorem provides regularity of the Galerkin truncated semiflow with bounds uniform in m under conditions (A) and (B1) analogous to the regularity result for the semiflow Φ in [15, Theorem 2.4].

Theorem 2.6 (Regularity of the Galerkin truncated semiflow, local version)**.** *Assume* (A) *and* (B1)*. Choose $R \in (0, \delta_*]$ small enough such that $\mathcal{D}_K^{-R} \neq \emptyset$ and pick $U^0 \in \mathcal{D}_K^{-R}$. Then there is $T_* = T_*(R, U^0) > 0$ and $m_*(R,U^0) \in \mathbb{N}$ such that for $m \geq m_*$ there exists a semiflow Φ_m^t of* (2.12) *of class*

$$\Phi_m \in \bigcap_{\substack{j+k\leq N\\ \ell\leq k\leq K}} \mathcal{C}_{\mathrm{b}}^{(\underline{j},\ell)}(B_{R/2}^{\mathcal{Y}_K}(U^0) \times [0,T_*]; \mathcal{B}_R^{\mathcal{Y}_{k-\ell}}(U^0))\,. \tag{2.20a}$$

The bounds on Φ_m and T_* depend only on the bounds afforded by (B1) *and* (2.1)*, on R, and on U^0. In particular,*

$$\Phi_m \in \mathcal{C}_{\mathrm{b}}^K(B_{R/2}^{\mathcal{Y}_K}(U^0) \times [0,T_*]; \mathcal{B}_R^{\mathcal{Y}}(U^0))\,. \tag{2.20b}$$

Proof. The proof is an application of Theorem A.9 (a) on contraction mappings on a scale of Banach spaces. We consider Π from (2.2) and write the corresponding contraction map for the semiflow Φ_m of the projected system as

$$\Pi_m(W;U,h) = \mathbb{P}_m \Pi(W;\mathbb{P}_m U,h)\,. \tag{2.21}$$

We replace N from Theorem A.9 by $N-1$, set $\mu = T$, $\mathcal{I} = (0,T_*)$, $\mathcal{X} = \mathcal{Y}_K$ with $\mathcal{U} \equiv \mathcal{U}_K = \mathcal{B}^{\mathcal{X}}_{R/2}(U^0)$, $w = W$, and $\mathcal{Z}_j = \mathcal{C}([0,1];\mathcal{Y}_j)$ with $\mathcal{W}_j = \mathcal{C}([0,1];\mathcal{B}^{\mathcal{Z}_j}_R(U^0))$ for $j = 0,\dots,K$. We now show that the contraction maps Π_m satisfy conditions (i) and (ii) of Theorem A.9 for some $T_*(R,U^0) > 0$ and $m \geq m_*(R,U^0)$. We first show that Π_m maps each $\mathcal{W}_0,\dots,\mathcal{W}_K$ into itself. We estimate, using (B1) and (2.1), that

$$\begin{aligned} \|\Pi_m(W;U,T) - U^0\|_{\mathcal{Y}_j} &\leq \|\mathrm{e}^{\tau T A}U^0 - U^0\|_{\mathcal{Y}_j} + \|\mathrm{e}^{\tau T A}(\mathbb{P}_m U^0 - U^0)\|_{\mathcal{Y}_j} \\ &\quad + \mathrm{e}^{\omega T}\,R/2 + T\,\mathrm{e}^{\omega T}\,M_j\,. \end{aligned} \tag{2.22}$$

Choosing $T_* = T_*(R,U^0) > 0$ sufficiently small, the second line of (2.22) can be made less than $3R/4$. Moreover, for a possibly smaller value of T_*, there exists $m_* = m_*(R,U^0)$ such that for all $m \geq m_*$, $T \in [0,T_*]$, and $\tau \in [0,1]$ the first line of (2.22) is less than $R/4$. Then, the right hand side of (2.22) is less than R which proves that Π_m maps back into $\mathcal{W}_j$. Assumption (B1) and (A) then imply condition (i) of Theorem A.9. To show condition (ii) we estimate, noting that $N > K$ by (B1), that

$$\|\mathrm{D}_W \Pi_m(W;U,T)\|_{\mathcal{E}(\mathcal{C}_{\mathrm{b}}([0,1];\mathcal{Y}_j))} \leq T\,\mathrm{e}^{\omega T}\,M_j'\,, \tag{2.23}$$

so that Π_m is a uniform contraction for all $m \geq m_*$, $U \in \mathcal{U}$, $W \in \mathcal{W}_j$, and $T \in \mathcal{I} = (0,T_*)$ for every $j = 0,\dots,K$ with a possibly smaller value of T_*.

Hence, Π_m satisfies conditions (i) and (ii) of Theorem A.9 with bounds and contraction constants which are uniform in $m \geq m_*$ so that Theorem A.9 (a) implies that Φ_m is of class (2.20a). The simplified special case (2.20b) is a direct consequence of Lemma A.2. □

Theorem 2.6 does not guarantee that m_* and T_* can be chosen uniformly over $\mathcal{D}$. The following theorem states that such uniformity can be obtained, however, over domains other than balls at the expense of stepping up on the scale of Hilbert spaces. The situation is analogous to that for the semiflow Φ; see [15, Theorem 2.6 and Remark 2.8].

Theorem 2.7 (Regularity of Galerkin truncated semiflow, uniform version)**.** *Assume* (A) *and* (B1). *Choose $\delta \in (0,\delta_*]$ small enough such that $\mathcal{D}^{-\delta}_{K+1} \neq \emptyset$. Then there exists $T_* = T_*(\delta) > 0$ and $m_*(\delta) \in \mathbb{N}$ such that for $m \geq m_*$ the semiflow $(U,t) \mapsto \Phi^t_m(U)$ of* (2.12) *satisfies* (2.20a) *with bounds which are uniform for all $U^0 \in \mathcal{D}^{-\delta}_{K+1}$ with $R = \delta$ and such that*

$$\Phi_m \in \bigcap_{\substack{j+k \leq N \\ \ell \leq k \leq K+1}} \mathcal{C}^{(\underline{j},\ell)}_{\mathrm{b}}(\mathcal{D}^{-\delta}_{K+1} \times [0,T_*];\mathcal{Y}_{k-\ell}) \tag{2.24a}$$

with bounds which are uniform in $m \geq m_*$. The bounds on Φ_m, m_* and T_* depend only on the bounds afforded by (B0) *rsp.* (B1), (2.1), *and* (2.16), *and on δ. Moreover, Φ_m maps into $\mathcal{D}_K$ and, when $N > K+1$,*

$$\Phi_m \in \mathcal{C}^{K+1}_{\mathrm{b}}(\mathcal{D}^{-\delta}_{K+1} \times [0,T_*];\mathcal{Y}) \tag{2.24b}$$

with corresponding uniform bounds.

Proof. We continue to work in the setting introduced in the proof of Theorem 2.6. Here, we need to verify that the conditions of Theorem A.9 are satisfied uniformly in $U^0 \in \mathcal{D}_{K+1}^{-\delta}$ for both Π_m and $\tilde{\Pi}_m$. First, due to (B1), each of the Π_m is well-defined as a map from $\mathcal{W}_j \times \mathcal{U} \times \mathcal{I}$ into $\mathcal{Z}_j$ for $U^0 \in \mathcal{D}_{K+1}^{-\delta}$ and has the required regularity. To show that there is $m_*(\delta)$ such that Π_m maps $\mathcal{W}_0, \dots, \mathcal{W}_K$ back into itself, we apply (2.22) for every $U^0 \in \mathcal{D}_{K+1}^{-\delta}$. We bound the first line on the right of (2.22) by

$$\begin{aligned} &\|\mathrm{e}^{\tau T A} U^0 - U^0\|_{\mathcal{Y}_j} + \|\mathrm{e}^{\tau T A}(\mathbb{P}_m U^0 - U^0)\|_{\mathcal{Y}_j} \\ &\qquad \le \max_{t \in [0,T]} \bigl(T \,\|A \mathrm{e}^{tA} U^0\|_{\mathcal{Y}_j} + \|\mathrm{e}^{tA} \mathbb{Q}_m U^0\|_{\mathcal{Y}_j} \bigr) \le \mathrm{e}^{\omega T} R_{K+1} \, (T + 1/m) \,, \end{aligned} \tag{2.25}$$

where R_{K+1} is defined in (2.16) and $j = 0, \dots, K$. Inserting this estimate into (2.22), we see that we can choose $T_* > 0$ small enough such that $\Pi_m(\,\cdot\,; U, T)$ maps $\mathcal{B}_R^{\mathcal{Y}_j}(U^0)$ with $R = \delta$ into itself for all $U^0 \in \mathcal{D}_{K+1}^{-\delta}$, $U \in \mathcal{U}$, $m \ge m_*$, $T \in [0, T_*]$, and $j = 0, \dots, K$. Hence, Π_m satisfies the conditions of Theorem A.9 with bounds which are uniform in $m \ge m_*$, $T \in (0, T_*)$, and $U^0 \in \mathcal{D}_{K+1}^{-\delta}$. This shows that (2.20a) holds uniformly for $U^0 \in \mathcal{D}_{K+1}^{-\delta}$ and $m \ge m_*(\delta)$.

Next, we show that

$$A\Phi_m \in \bigcap_{\substack{j+k \le N \\ \ell \le k \le K}} \mathcal{C}_{\mathrm{b}}^{(\underline{j}, \ell)}(\mathcal{D}_{K+1}^{-\delta} \times [0, T_*]; \mathcal{Y}_{k-\ell}) \tag{2.26}$$

with uniform bounds in $m \ge m_*$. Consider the linear fixed point equation $\tilde{W}_m = \tilde{\Pi}_m(\tilde{W}_m; U, T)$ with

$$\begin{aligned} \tilde{\Pi}_m(\tilde{W}_m; U, T)(\tau) &= \mathrm{e}^{\tau T A} \left(A \mathbb{P}_m U + B(W_m(0)) \right) - B(W_m(\tau)) \\ &\quad + T \int_0^\tau \mathrm{e}^{(\tau - \sigma) T A} \, \mathrm{D}B_m(W_m(\sigma)) (\tilde{W}_m(\sigma) + B(W_m(\sigma)) \, \mathrm{d}\sigma \end{aligned} \tag{2.27}$$

where $W_m(U, T)(\tau) = \Phi_m^{\tau T}(U)$. Integrating the right hand side of (2.2) by parts, replacing B with B_m and U^0 by $\mathbb{P}_m U^0$ we see that the fixed point $\tilde{W}_m$ of $\tilde{\Pi}_m$ satisfies $\tilde{W}_m = A W_m$ in $\mathcal{Z}_j$ for $j = 0, \dots, K$. We consider $\tilde{\Pi}_m$ with $\mathcal{U}$, $\mathcal{Z}_j$ and $\mathcal{I}$ as before, and set $\mathcal{W}_j = \mathcal{C}([0,1]; \mathcal{B}_r^{\mathcal{Z}_j}(0))$ with $r > 0$ large enough that $\tilde{\Pi}_m(\cdot, U, T)$ maps $\mathcal{W}_j$ into itself for $m \ge m_*$, $U \in \mathcal{U}$, $T \in \mathcal{I}$. Since Φ_m is of class (2.20a), Lemma A.6 (a) and Lemma A.8 (a) imply that $\tilde{\Pi}_m$ satisfies the conditions of Theorem A.9 with N replaced by $N - 2$. Therefore Theorem A.9 (a) applies and proves (2.26).

Moreover, $B_m \circ \Phi_m$ is of class (2.26) with uniform bounds for $m \ge m_*$ due to the chain rule, Lemma A.6 (a) and the fact that Φ_m is of class (2.20a) with uniform bounds in U^0 and $m \ge m_*$. We conclude that $\partial_t \Phi_m = A\Phi_m + B_m \circ \Phi_m$ is also of class (2.26) with uniform bounds for $m \ge m_*$.

Finally, as both $A\Phi_m$ and $\partial_t \Phi_m$ are of class (2.26), Lemma A.4 implies (2.24a). The simplified special case (2.24b) is a direct consequence of Lemma A.2. □

2.4. **Accuracy of derivatives of Galerkin truncated semiflow.** In this section, we consider how the perturbation of the contraction map Π introduced by the projection of the evolution equation (1.1) onto the subspace $\mathbb{P}_m\mathcal{Y}$ propagates into derivatives of the resulting semiflow.

As before, we consider a local and a uniform version of each result; the scales we use are defined, separately for the two cases, as follows. In the local version, we follow the setting of Theorem 2.6, where we consider initial data

$$U \in \mathcal{U} \equiv \mathcal{B}^{\mathcal{X}}_{R_*}(U^0) \qquad \text{where} \qquad \mathcal{X} = \mathcal{Y}_K \,. \tag{2.28a}$$

The semiflows are considered as maps

$$\Phi^t, \Phi^t_m \colon \mathcal{B}^{\mathcal{Y}_K}_{R_*}(U^0) \to \mathcal{Y}_j \quad \text{for} \quad j = 0, \dots, K, \tag{2.28b}$$

where $m \geq m_*(\delta, U^0)$ and $R_* = R/2$.

In the uniform version, we follow the setting of Theorem 2.7 where we consider initial data

$$U \in \mathcal{U} = \mathcal{D}^{-\delta}_{K+1} \subset \mathcal{X} \equiv \mathcal{Y}_{K+1} \,. \tag{2.29a}$$

The semiflows are considered as maps

$$\Phi^t, \Phi^t_m \colon \mathcal{D}^{-\delta}_{K+1} \to \mathcal{Y}_j \quad \text{for} \quad j = 0, \dots, K+1, \tag{2.29b}$$

for some fixed $\delta > 0$ where $m \geq m_*(\delta)$.

To handle the complexity of these estimates it is useful to define norms on the various objects that contain all combinatorially possible orders of differentiation and scale rungs subject to certain relevant side constraints. The need to consider such norms arises through the implicit nature of the definition of the semiflow and the use of the chain rule. Here, any attempt to estimate a particular derivative on a particular rung of the scale will produce terms of all intermediate orders of differentiation and scale rungs. We therefore estimate all derivatives at once.

We have to deal with two different types of objects: contraction maps which are functions of three arguments whose corresponding norms are denoted $\|\!| \cdot |\!\|$ and semiflows which are functions of two arguments whose corresponding norms are denoted $\| \cdot \|$.

In our setting, there are two natural global parameters, namely N, the degree of differentiability of the nonlinearity, and K, the number of rungs on our scale as defined in condition (B1). Two more characteristic parameters are needed. First, the *loss index* S which forces the image of the map be estimated at least S rungs down the scale. We will see that a loss of S scale rungs translates into $O(m^{-S})$-smallness of the perturbation caused by the projector $\mathbb{P}_m$. Second, a *lowest rung index* L which forces the estimation of the image of the map to occur at least L rungs up from the bottom of the scale. This leads us to define a four parameter family of norms for functions $\Pi = \Pi(w; u, \mu)$ mapping $\mathcal{W}_{k+S} \times \mathcal{U} \times \mathcal{I}$ to $\mathcal{Z}_{k-\ell}$,

$$|\!|\!|\Pi|\!|\!|_{N,K,L,S} = \max_{\substack{i+j+k \leq N-S \\ L+\ell \leq k \leq K-S}} \|\mathrm{D}^i_w \mathrm{D}^j_u \partial^\ell_\mu \Pi\|_{\mathcal{L}_\infty(\mathcal{W}_{k+S} \times \mathcal{U} \times \mathcal{I}; \mathcal{E}^i(\mathcal{Z}_{k+S}, \mathcal{E}^j(\mathcal{X}; \mathcal{Z}_{k-\ell})))} \tag{2.30a}$$

for $0 \leq L \leq K - S \leq N - S$. When studying semiflows, we identify $w = W$, $u = U$, $\mu = T$, $\mathcal{I} = (0, T_*)$, $\mathcal{Z}_j = \mathcal{C}([0,1]; \mathcal{Y}_j)$, $\mathcal{U} = \mathcal{B}^{\mathcal{Y}_j}_{R/2}(U^0)$, $\mathcal{X} = \mathcal{Y}_K$, and $\mathcal{W}_j = \mathcal{C}([0,1]; \mathcal{B}^{\mathcal{Y}_j}_R(U^0))$, and use Π defined by (2.2). We abbreviate

$$|\!|\!|\Pi|\!|\!|_{N,K,L} = |\!|\!|\Pi|\!|\!|_{N,K,L,0} \,, \tag{2.30b}$$

$$|\!|\!|\Pi|\!|\!|_{N,K} = |\!|\!|\Pi|\!|\!|_{N,K,0,0} \,. \tag{2.30c}$$

Functions $w = w(u,\mu)$ mapping $\mathcal{U} \times \mathcal{I}$ to $\mathcal{Z}_j$ are equipped with the three parameter family of norms

$$\|w\|_{N,K,L} = \max_{\substack{j+k\leq N \\ L+\ell\leq k\leq K}} \|\mathrm{D}_u^j \partial_\mu^\ell w\|_{\mathcal{L}_\infty(\mathcal{U}\times\mathcal{I};\mathcal{E}^j(\mathcal{X};\mathcal{Z}_{k-\ell}))} \tag{2.30d}$$

for $0 \leq L \leq K \leq N$, where we abbreviate

$$\|w\|_{N,K} = \|w\|_{N,K,0} \,. \tag{2.30e}$$

With this notation, a function $(u,\mu) \mapsto \Pi(u,\mu)$ which does not depend on w satisfies

$$|\!|\!|\Pi|\!|\!|_{N,K,L,S} = \|\Pi\|_{N-S,K-S,L} \,. \tag{2.31}$$

The next pair of results concerns the stability of the semiflow and its derivatives under spectral truncation.

Theorem 2.8 (Projection error for the semiflow, local version)**.** *Assume* (A) *and* (B1) *and $R \in (0, \delta_*]$ small enough such that $\mathcal{D}_K^{-R} \neq \emptyset$ and pick $U^0 \in \mathcal{D}_K^{-R}$. Let $T_* = T_*(R, U^0)$ and $m_* = m_*(R, U^0)$ be as in Theorem 2.6. Then, for every $0 \leq P \leq K$,*

$$\|\Phi - \Phi_m\|_{N-P-1,K-P} = O(m^{-P}) \tag{2.32}$$

where the norm in (2.32) *is defined with respect to the spaces* (2.28)*. The order constants depend only on the bounds afforded by* (B1) *and* (2.1)*, on U^0, and on R.*

Proof. We apply Theorem A.9 to obtain a bound on $\|\Phi - \Phi_m\|_{N-P-1,K-P}$ in terms of $|\!|\!|\Pi - \Pi_m|\!|\!|_{N-1,K,0,P}$, with $\mathcal{Z}_k$, $\mathcal{X}$ etc. specified above. We already verified conditions (i) and (ii) of Theorem A.9 in the proof of Theorem 2.6. Thus, in order to prove (2.32), it suffices to show that

$$|\!|\!|\Pi - \Pi_m|\!|\!|_{N-1,K,0,P} = O(m^{-P}) \,. \tag{2.33}$$

Writing

$$(\Pi - \Pi_m)(W; U, T)(\tau) = G_m(U,T)(\tau) + I_m(W,T)(\tau) \tag{2.34}$$

with

$$G_m(U,T)(\tau) = \mathbb{Q}_m \mathrm{e}^{T\tau A} U \,,$$
$$I_m(W,T)(\tau) = \mathbb{Q}_m \int_0^\tau T\mathrm{e}^{(\tau-\sigma)TA} B(W(\sigma))\,\mathrm{d}\sigma \,,$$

we apply Lemma A.6 to both terms on the right of (2.34) in different ways. For the first term, since G_m does not depend on W, we can apply Lemma A.6 with $\Pi = \mathrm{id}$, $\Sigma = \mathbb{P}_m$, and $v(U,T)(\tau) = w(U,T)(\tau) = \mathrm{e}^{\tau TA} U$, so that there exists $c_1 > 0$ such that

$$|\!|\!|G_m|\!|\!|_{N-1,K,0,P} = \|G_m\|_{N-1-P,K-P} \leq c_1 \, |\!|\!|\mathbb{Q}_m|\!|\!|_{N-1,K,0,P} = O(m^{-P}) \,,$$

where the first equality is due to (2.31). Here, and further below, we implicitly make use of estimate (2.1) on the bound of the linear semigroup and estimate (2.11) on the Galerkin remainder.

To apply Lemma A.6 to the second term on the right of (2.34), we identify N and K there with $N-1$ and an arbitrary $\kappa \in P, \ldots, K$ here. Then, setting $\Pi = \mathrm{id}$, $\Sigma = \mathbb{P}_m$, $u = W \in \mathcal{U}_\kappa \equiv \mathcal{C}([0,1]; \mathcal{B}_R^{\mathcal{Y}_\kappa}(U^0))$, and

$$v(u,T)(\tau) = w(u,T)(\tau) = \int_0^\tau T\mathrm{e}^{(\tau-\sigma)TA} B(W(\sigma))\,\mathrm{d}\sigma \,,$$

Lemma A.6 asserts that there exists $c_2 > 0$ such that

$$\|I_m\|_{N-1-P,\kappa-P} \leq c_2 \,\|\mathbb{Q}_m\|_{N-1,\kappa,0,P} = O(m^{-P})\,.$$

Then, by Lemma A.5, with N replaced by $N-1$ and $S = P$,

$$\|I_m\|_{N-1,K,0,P} = O(m^{-P})\,.$$

The constants c_1 and c_2 depend only on the bounds on B from (B1) and the bounds from (2.1). Altogether, this verifies (2.33), thus concludes the proof. □

Theorem 2.9 (Projection error for the semiflow, uniform version)**.** *Assume* (A) *and* (B1) *with* $N > K+1$*, and let* $\delta \in (0, \delta_*]$ *small enough such that* $\mathcal{D}^{-\delta}_{K+1}$ *is nonempty. Let* $T_* = T_*(\delta) > 0$ *and* $m_* = m_*(\delta)$ *be as in Theorem 2.7. Then, for every* $0 \leq P \leq K+1$*,*

$$\|\Phi - \Phi_m\|_{N-P-1,K+1-P} = O(m^{-P})\,, \tag{2.35}$$

where the norm in (2.35) *is defined with respect to the spaces* (2.29)*. The order constants depend only on* δ*, and on the bounds afforded by* (B1)*,* (2.1)*, and* (2.16)*.*

Proof. First, we show that

$$\|\Phi - \Phi_m\|_{N-P-1,K-P} = O(m^{-P})\,, \tag{2.36}$$

for $0 \leq P \leq K$ on the scale $\{\mathcal{Z}_j\}_{j=0,\dots,K}$. To this end, define Π and Π_m, $\mathcal{U}$, $\mathcal{W}_j$ etc., with $R = \delta$ as in the proof of Theorem 2.8.

We have already shown in the proof of Theorem 2.7 that conditions (i) and (ii) of Theorem A.9 hold uniformly in $m \geq m_*$ and $U^0 \in \mathcal{D}^{-\delta}_{K+1}$. Moreover, (2.33) holds true uniformly in $U^0 \in \mathcal{D}^{-\delta}_{K+1}$. This is easily verified by checking that each of the estimates in the proof of Theorem 2.8 holds uniformly under the conditions of Theorem 2.9. Hence, Theorem A.9 (b) implies (2.36).

Next, we apply Theorem A.9 to obtain a bound on $\|\tilde\Pi - \tilde\Pi_m\|_{N-2,K,0,P}$, where $\tilde\Pi_m$ is from (2.27) and $\tilde\Pi$ is defined correspondingly. We have shown in the proof of Theorem 2.7 that $\tilde\Pi_m$ (and hence also $\tilde\Pi$) satisfy the conditions of Theorem A.9 uniformly for $m \geq m_*$. Estimating each term of the corresponding analogue to (2.34) via Lemma A.6 and Lemma A.8, we find that $\|\tilde\Pi - \tilde\Pi_m\|_{N-2,K,0,P} = O(m^{-P})$. Then, Theorem A.9 (b) implies that $\|A\Phi - A\Phi_m\|_{N-P-2,K-P} = O(m^{-P})$ so that, for $0 \leq P \leq K$,

$$\|\Phi - \Phi_m\|_{N-P-1,K-P+1,1} = O(m^{-P})\,. \tag{2.37}$$

Finally, we prove that

$$\|\partial_t\Phi - \partial_t\Phi_m\|_{N-P-2,K-P} = O(m^{-P})\,. \tag{2.38}$$

We note that $\partial_t(\Phi - \Phi_m) = A(\Phi - \Phi_m) + B\circ\Phi - B_m \circ \Phi_m$, where the required bound on the first term on the right has already been established. For the second term, we use Lemma A.6 with $\Pi = B$, $\Sigma = B_m$, $\mu = t \in \mathcal{I} = (0, T_*)$, $u = U^0 \in \mathcal{U} = \mathcal{D}^{-\delta}_{K+1}$, $w(U^0, t) = \Phi^t(U^0)$, $v(U^0, t) = \Phi^t_m(U^0)$, $\mathcal{X} = \mathcal{Y}_{K+1}$, $\mathcal{Z}_j = \mathcal{Y}_j$, and $\mathcal{W}_j = \mathcal{D}_j$ for $j = 0, \dots, K$. Hence, there exists a constant c_1 such that

$$\|B\circ\Phi - B_m \circ \Phi_m\|_{N-2-P,K-P} \leq c_1 \,\|\mathbb{Q}_m B\|_{N-2,K,0,P} + c_1 \,\|\Phi - \Phi_m\|_{N-2-P,K-P}\,.$$

To estimate $\|\mathbb{Q}_m B\|_{N-2,K,0,P}$, we apply Lemma A.6 for each $\kappa \in P, \dots, K$ with $\Pi = \mathrm{id}$, $\Sigma = \mathbb{P}_m$, $u = W$, $v(u,h) = w(u,h) = B(W)$, $\mathcal{U}$ replaced by $\mathcal{U}_\kappa \equiv \mathcal{W}_\kappa$, N

replaced by $N-1$, and K replaced by κ. Hence, there is some constant c_2 such that

$$\|\mathbb{Q}_m B(W)\|_{N-2-P,\kappa-P} \leq c_2 \,\|\mathbb{Q}_m\|_{N-2,\kappa,0,P} = O(m^{-P})\,.$$

Then, Lemma A.5 with N replaced by $N-1$ and $S=P$ implies

$$\|\mathbb{Q}_m B\|_{N-2,K,0,P} = O(m^{-P})\,.$$

Altogether, this proves (2.38). Finally, (2.35) follows from Lemma A.4 with $L=0$ due to (2.36), (2.37), and (2.38). $\square$

3. A-stable Runge–Kutta methods under Galerkin truncation

In this section, we study a class of A-stable Runge–Kutta methods which are well-defined when applied to the semilinear PDE (1.1) under assumptions (A) and (B1). We prove regularity of spectral Galerkin approximations of such methods uniformly in the spatial resolution and derive estimates for the approximation error. The class of methods we consider is the same as in [15].

Applying an s-stage Runge–Kutta method to the semilinear evolution equation (1.1), we obtain

$$W = U^0\,\mathbb{1} + h\,\mathsf{a}\left(AW + B(W)\right), \tag{3.1a}$$

$$U^1 = U^0 + h\,\mathsf{b}^T\left(AW + B(W)\right). \tag{3.1b}$$

For $U \in \mathcal{Y}$ we write

$$\mathbb{1}\,U = \begin{pmatrix} U \\ \vdots \\ U \end{pmatrix} \in \mathcal{Y}^s\,, \quad W = \begin{pmatrix} W^1 \\ \vdots \\ W^s \end{pmatrix}\,, \quad B(W) = \begin{pmatrix} B(W^1) \\ \vdots \\ B(W^s) \end{pmatrix}\,,$$

where $W^1, \dots, W^s$ are the stages of the Runge–Kutta method,

$$(\mathsf{a}W)^i = \sum_{j=1}^s \mathsf{a}_{ij}\, W^j\,, \qquad \mathsf{b}^T W = \sum_{j=1}^s \mathsf{b}_j\, W^j\,,$$

and A acts diagonally on the stages, i.e., $(AW)^i = AW^i$ for $i=1,\dots,s$.

A more suitable form, required later, is achieved by rewriting (3.1a) as

$$W = \Pi(W;U,h) \equiv (\mathrm{id} - h\mathsf{a}A)^{-1}\,(\mathbb{1}U + h\mathsf{a}B(W)) \tag{3.2}$$

and

$$\Psi^h(U) = \mathsf{S}(hA)U + h\mathsf{b}^T\,(\mathrm{id} - h\mathsf{a}A)^{-1}\,B(W(U,h))\,, \tag{3.3}$$

where S is the so-called *stability function*

$$\mathsf{S}(z) = 1 + z\mathsf{b}^T\,(\mathrm{id} - z\mathsf{a})^{-1}\,\mathbb{1}\,. \tag{3.4}$$

We now make a number of assumptions on the method and its interaction with the linear operator A. First, we assume that the method is A-stable in the sense of [12]. Setting $\mathbb{C}^- = \{z \in \mathbb{C} \colon \operatorname{Re} z \leq 0\}$, the conditions are as follows.

(RK1) The stability function (3.4) is bounded with $|\mathsf{S}(z)| \leq 1$ for all $z \in \mathbb{C}^-$.
(RK2) The $s \times s$ matrices $\mathrm{id} - z\mathsf{a}$ are invertible for all $z \in \mathbb{C}^-$.

Sometimes, we will also assume that a is invertible. Gauss–Legendre Runge–Kutta methods satisfy conditions (RK1) and (RK2) with a invertible [15, Lemma 3.6].

We now summarize the analytic properties of the operators appearing in (3.2) and (3.3), where we use the convention $\|W\|_{\mathcal{Y}^s_\ell} = \max_{j=1}^s \|W^j\|_{\mathcal{Y}_\ell}$. Proofs can be found in [15, Section 3.2].

Lemma 3.1. *Assume* (RK1)*,* (RK2)*, and* (A)*. Then there exist $h_* > 0$, $\Lambda \geq 1$, $\sigma \geq 0$, and $c_{\mathsf{S}} \geq 1$ such that*

$$\|(\mathrm{id} - h\mathsf{a}A)^{-1}\|_{\mathcal{Y}^s \to \mathcal{Y}^s} \leq \Lambda\,, \tag{3.5a}$$

$$\|\mathsf{S}(hA)\|_{\mathcal{Y}^s \to \mathcal{Y}^s} \leq 1 + \sigma\, h \leq c_{\mathsf{S}}\,, \tag{3.5b}$$

$$\|h\mathsf{a}A(\mathrm{id} - h\mathsf{a}A)^{-1}\|_{\mathcal{Y}^s \to \mathcal{Y}^s} \leq 1 + \Lambda\,, \tag{3.5c}$$

for all $h \in [0, h_*]$. Moreover, for any $\ell, n, \in \mathbb{N}_0$,

$$(W, h) \mapsto (\mathrm{id} - h\mathsf{a}A)^{-1} W \text{ is a map of class } \mathcal{C}_{\mathrm{b}}^{(\underline{n},\ell)}(\mathcal{Y}^s_\ell \times [0, h_*]; \mathcal{Y}^s)\,, \tag{3.6a}$$

$$(W, h) \mapsto h\mathsf{a}A(\mathrm{id} - h\mathsf{a}A)^{-1} W \text{ is a map of class } \mathcal{C}_{\mathrm{b}}^{(\underline{n},\ell)}(\mathcal{Y}^s_\ell \times [0, h_*]; \mathcal{Y}^s)\,, \tag{3.6b}$$

$$(W, h) \to h(\mathrm{id} - h\mathsf{a}A)^{-1} W \text{ is a map of class } \mathcal{C}_{\mathrm{b}}^{(\underline{n},\ell+1)}(\mathcal{Y}^s_\ell \times [0, h_*]; \mathcal{Y}^s)\,, \tag{3.6c}$$

and

$$(U, h) \mapsto \mathsf{S}(hA) U \text{ is a map of class } \mathcal{C}_{\mathrm{b}}^{(\underline{n},\ell)}(\mathcal{Y}_\ell \times [0, h_*]; \mathcal{Y})\,. \tag{3.6d}$$

3.1. **Regularity of Galerkin truncated time-discretization.** Let $W_m(U^0, h)$ denote the stage vector, $W_m^j(U^0, h)$ for $j = 1, \dots, s$ its components, and $\Psi_m^h(U^0, h)$ denote the numerical time-h map obtained by applying an s-stage Runge–Kutta method to the projected semilinear evolution equation (2.12) with initial value $u_m(0) = \mathbb{P}_m U^0$. Their regularity, with uniform bounds in the spatial resolution m, is stated in the following theorems which, again in local and uniform version, provide the analogue to what is known for the time-h map Ψ in the time-semidiscrete case [15, Theorems 3.14, 3.15, and Remark 3.17].

Theorem 3.2 (Regularity of Galerkin truncated numerical method, local version)**.** *Assume that the semilinear evolution equation* (1.1) *satisfies conditions* (A) *and* (B1)*. Apply a Runge–Kutta method Ψ subject to conditions* (RK1) *and* (RK2) *to it. Choose $R \in (0, \delta_*]$ such that $\mathcal{D}_K^{-R} \neq \emptyset$ and pick $U^0 \in \mathcal{D}_K^{-R}$. Let $R_* = R/(2\max\{c_{\mathsf{S}}, \Lambda\})$ with Λ and c_{S} from* (3.5)*. Then there exists $m_* = m_*(R, U^0)$ and $h_* = h_*(R, U^0) > 0$, such that for $m \geq m_*$ there exists a stage vector W_m whose components W_m^i as well as the numerical time-h map $\Psi_m^h(U, h) = \Psi_m^h(U)$ are of class* (2.20a) *with T_* replaced by h_*. The bounds on W_m, Ψ_m and h_* are independent of m and depend only on the bounds afforded by* (B1) *and* (3.5)*, on the coefficients of the method, R, and U^0.*

Proof. As in the proof of Theorem 2.6 on the regularity of the semiflow Φ_m, we apply Theorem A.9 (a) on contraction mappings on a scale of Banach spaces. Here we set $w = W$, $\mathcal{Z}_j = \mathcal{Y}_j^s$, and $\mathcal{W}_j = \mathcal{B}_R^{\mathcal{Y}_j^s}(\mathbb{1}U^0)$ for $j = 0, \dots, K$. We further identify $\mu = h$, $\mathcal{I} = (0, h_*)$, and $\mathcal{X} = \mathcal{Y}_K$ with $\mathcal{U} \equiv \mathcal{U}_K = \mathcal{B}_{R_*}(U^0) \subset \mathcal{Y}_K$.

The map Π for the stage vector W is defined by (3.2); we write the corresponding map $\Pi_m(W, U, h) = \mathbb{P}_m \Pi(W, \mathbb{P}_m U, h)$, analogous to (2.21) for the semiflow. We now show that the differentiability assumption on B is such that Π_m satisfies the conditions of Theorem A.9 for $m \geq m_*$ with a suitable choice of m_*.

First, we show that Π_m maps each $\mathcal{W}_0, \dots, \mathcal{W}_K$ into itself uniformly for $h \in (0, h_*)$ and $U \in \mathcal{U}$. By Lemma 3.1 we estimate, for $W \in \mathcal{W}_j$,

$$\begin{aligned}\|\Pi_m(W; U, h) - \mathbb{1}U^0\|_{\mathcal{Y}_j^s} &\leq \|(\mathrm{id} - (\mathrm{id} - h\mathsf{a}A)^{-1})\mathbb{1}U^0\|_{\mathcal{Y}_j^s} + \|(\mathrm{id} - h\mathsf{a}A)^{-1}\mathbb{1}\mathbb{Q}_m U^0\|_{\mathcal{Y}_j^s} \\ &\quad + \Lambda\, R_* + h\,\Lambda\,\|\mathsf{a}\|\, M_j\,. \end{aligned} \tag{3.7}$$

Using Lemma 3.1, we can find $h_*(R, U^0)$ and $m_*(R, U^0)$ such that for $m \geq m_*$ and $h \in (0, h_*)$ the first line on the right of (3.7) is less than $R/4$. By possibly shrinking h_* further, the second line is less than $3R/4$, so that $\Pi_m(\,\cdot\,; U, h)$ maps $\mathcal{W}_j$ into itself. Second, assumptions (B1) and Lemma 3.1 ensure that Π_m satisfies condition (i) of Theorem A.9 for $m \geq m_*$. The contraction estimate, condition (ii) of Theorem A.9, follows from

$$\begin{aligned}\|\mathrm{D}_W \Pi_m(W; U, h)\|_{\mathcal{Y}_j^s \to \mathcal{Y}_j^s} &\leq h\,\|(\mathrm{id} - h\mathsf{a}A)^{-1}\mathsf{a}\|_{\mathcal{Y}_j^s \to \mathcal{Y}_j^s}\, \|\mathrm{D}B_m(W)\|_{\mathcal{Y}_j^s \to \mathcal{Y}_j^s} \\ &\leq h\,\Lambda\,\|\mathsf{a}\|\, M_j' \end{aligned} \tag{3.8}$$

for $j = 0, \dots, K$. Thus, by possibly shrinking h_* again, the right hand bound can be made less than 1, and condition (ii) is met for $m \geq m_*$. Thus Theorem A.9 (a) implies that W_m^j is of class (2.20a) for $j = 1, \dots, s$. The same holds true for Ψ_m due to the chain rule on scales of Banach spaces, Lemma A.6 (a), applied to (3.3) using Lemma 3.1. □

As for the semiflow Φ_m, there is also a uniform version of this result.

Theorem 3.3 (Regularity of Galerkin truncated time-discretization, uniform version)**.** *Assume* (A) *and* (B1)*, as well as* (RK1) *and* (RK2)*. Pick $\delta \in (0, \delta_*]$ such that $\mathcal{D}_{K+1}^{-\delta}$ is nonempty. Then there is $h_* = h_*(\delta) > 0$ and $m_* = m_*(\delta)$ such that for $m \geq m_*$ the statements of Theorem 3.2 hold true with $R = \delta$ and bounds which are uniform in $U^0 \in \mathcal{D}_{K+1}^{-\delta}$ and $m \geq m_*$. Moreover, for $m \geq m_*$, the components W_m^j of the stage vector $W_m(U, h)$ are of class* (2.24a) *with T_* replaced by h_*, and, if the Runge–Kutta matrix a is invertible, the numerical time-h map Ψ_m is also of class* (2.24a) *with T_* replaced by h_*. The bounds on W_m, Ψ_m and h_* are independent of $m \geq m_*$ and only depend on the bounds afforded by* (B1)*,* (2.16)*, and* (3.5)*, on the coefficients of the method, and on δ.*

Proof. Let $\mathcal{Z}_j = \mathcal{Y}_j$, $\mathcal{W}_j = \mathcal{B}_R^{\mathcal{Y}_k^s}(\mathbb{1}U^0)$, and $\mathcal{U} = \mathcal{B}_{R_*}^{\mathcal{Y}_K}(U^0)$ for $j = 0, \dots, K$ as in the proof of Theorem 3.2, taking $R = \delta$ and R_* as in Theorem 3.2. First, due to (B1), the map Π_m is well-defined from $\mathcal{W}_j \times \mathcal{U} \times \mathcal{I}$ into $\mathcal{Z}_j$ with the required regularity properties and with bounds that are uniform in m and $U^0 \in \mathcal{D}_{K+1}^{-\delta}$. To show that Π_m maps $\mathcal{W}_j$ back into $\mathcal{W}_j$ for $m \geq m_*(\delta)$ with a suitable choice of m_*, note that, for $j = 0, \dots, K$,

$$
\begin{aligned}
&\|(\mathrm{id} - (\mathrm{id} - h\mathsf{a}A)^{-1}\mathbb{1})U^0\|_{\mathcal{Y}_j^s} + \|(\mathrm{id} - h\mathsf{a}A)^{-1}\mathbb{1}\mathbb{Q}_m U^0\|_{\mathcal{Y}_j^s} \\
&\quad \le h \max_{s\in[0,h]} \|\mathsf{a}A(\mathrm{id} - s\mathsf{a}A)^{-2}\mathbb{1}U^0\|_{\mathcal{Y}_j^s} + \Lambda\,\|\mathbb{1}\mathbb{Q}_m U^0\|_{\mathcal{Y}_j^s} \\
&\quad \le (h\,\Lambda^2\,\|\mathsf{a}\| + \Lambda/m) \sup_{U\in\mathcal{D}_{K+1}^{-\delta}} \|U\|_{\mathcal{Y}_j} \\
&\quad \le (h\,\Lambda^2\,\|\mathsf{a}\| + \Lambda/m)\, R_{K+1}\,,
\end{aligned} \tag{3.9}
$$

where R_{K+1} is defined by (2.16). Inserting this estimate into (3.7), we see that we can choose $h_*(\delta) > 0$ small enough and $m_*(\delta)$ big enough such that $\Pi_m(\,\cdot\,; U, h)$ maps each $\mathcal{W}_j$ into itself and, due to (3.8), such that Π_m is a contraction on each $\mathcal{W}_j$ uniformly for $U^0, U \in \mathcal{D}_{K+1}^{-\delta}$, and $h \in [0, h_*]$. So the conditions of Theorem A.9 are satisfied uniformly for $m \ge m_*$, $h \in (0, h_*)$, and $U^0 \in \mathcal{D}_{K+1}^{-\delta}$.

Applying A to $\Pi_m(W, U, h)$ yields

$$
AW_m = A\Pi_m(W_m, \mathbb{P}_m U, h) = (\mathrm{id} - h\mathsf{a}A)^{-1}\mathbb{1}A\mathbb{P}_m U + h\mathsf{a}A(\mathbb{1}U - h\mathsf{a}A)^{-1}\mathbb{P}_m B(W_m)\,.
$$

Using Lemma A.6 (a), the chain rule on a scale of Banach spaces, together with the estimates of Lemma 3.1, we find that AW_m^j is of class (2.26) for $j = 1, \dots, s$. Moreover, on the $(K+1)$-scale $\mathcal{Z}_j = \mathcal{Y}_j$ for $j = 0, \dots, K$ and $\mathcal{Z}_{K+1} = \mathcal{Y}_K$ with $\mathcal{W}_j = \mathcal{D}_j^s$ for $j = 0, \dots K$ and $\mathcal{W}_{K+1} = \mathcal{D}_K^s$, and with $\mathcal{U} = \mathcal{D}_{K+1}^{-\delta}$, $\mathcal{X} = \mathcal{Y}_{K+1}$, and $\mathcal{I} = (0, h_*)$, the map Π_m satisfies conditions (i) and (ii) of Theorem A.9. (Here we have used Lemma 3.1, in particular (3.6c), once again.) Therefore, $\partial_h W_m^j$ is of class (2.26). Then Lemma A.4 implies that W_m^j is of class (2.24a). If a is invertible, we can use (3.3), (3.6b) and the chain rule in the form of Lemma A.6 (a) to show that Ψ_m is also of class (2.24a). □

Remark 3.4. When a is not invertible, an according modification of the proof of Theorem 3.3 yields the weaker statement

$$
\Psi_m \in \bigcap_{\substack{j+k\le N-1 \\ k\le K}} \mathcal{C}_{\mathrm{b}}^{(j,k+1)}(\mathcal{D}_{K+1}^{-\delta} \times [0, T_*]; \mathcal{D}) \tag{3.10}
$$

for $m \ge m_*$ with bounds that depend only on the bounds afforded by (B1), (2.16), and (3.5), on the coefficients of the method, and on δ. An analogous statement holds true for Ψ [15, Remark 3.22].

3.2. **Convergence of Galerkin truncated time discretization.** Next, we prove a convergence result for the time-semidiscretization of the projected system. The error bounds are uniform in the spatial truncation parameter m.

Theorem 3.5 (Convergence of time discretization of projected system)**.** *Assume that the semilinear evolution equation* (1.1) *satisfies condition* (A)*, and apply a Runge–Kutta method of classical order p subject to conditions* (RK1) *and* (RK2) *to it. Assume further that* (B1) *holds with $K \ge p$. Pick $\delta \in (0, \delta_*]$ such that $\mathcal{D}_{p+1}^{-\delta}$ is non-empty, and fix $T > 0$. Then there exist positive constants h_*, m_*, c_1, and c_2 which only depend on the bounds afforded by* (B1), (3.5)*, on the coefficients of the method, and on δ, such that for every U^0 satisfying*

$$
\{\Phi^t(U^0) \colon t \in [0, T]\} \subset \mathcal{D}_{p+1}^{-\delta}\,, \tag{3.11}
$$

$h \in [0, h_*]$, and $m \geq m_*$, the numerical solution $(\Psi_m^h)^n(U^0)$ lies in $\mathcal{D}$ and satisfies

$$\|(\Psi_m^h)^n(U^0) - \Phi_m^{nh}(U^0)\|_{\mathcal{Y}} \leq c_2\, \mathrm{e}^{c_1 nh}\, h^p \tag{3.12}$$

so long as $nh \leq T$.

Proof. Convergence of the time semidiscretization under the above assumptions can be proved by a standard Gronwall argument, see [15, Theorem 3.24] where condition (A1) there is always satisfied in the setting here; it is stated as (3.5b) in Lemma 3.1. Then [15, Theorem 3.24] asserts that whenever a semiflow Φ satisfies (3.11), there exist constants c_1 and c_2 which only depend on the bounds afforded by (B1), (3.5), on the coefficients of the method, and on $\mathcal{D}_{p+1}^{-\delta}$ such that

$$\|(\Psi^h)^n(U^0) - \Phi^{nh}(U^0)\|_{\mathcal{Y}} \leq c_2\, \mathrm{e}^{c_1 nh}\, h^p \tag{3.13}$$

so long as $nh \leq T$.

Here, we need to apply this result with Φ replaced by Φ_m and Ψ by Ψ_m and show that we obtain uniform bounds in $m \geq m_*$ and U^0 satisfying (3.11). So we have to show that if condition (3.11) holds, then there is an analogue of this condition for the truncated system which is valid for all $m \geq m_*$ In other words, we have to find a $\mathcal{Y}_{p+1}$-bounded set $\tilde{\mathcal{D}}_{p+1} \subset \mathcal{D}_p \cap \mathcal{Y}_{p+1}$ and $\tilde{\delta} > 0$ such that

$$\{\Phi_m^t(U^0) \colon t \in [0,T]\} \subset \tilde{\mathcal{D}}_{p+1}^{-\tilde{\delta}} \tag{3.14}$$

holds for all U^0 satisfying (3.11) and all $m \geq m_*$. Applying Corollary 2.4 with $\mathcal{Y}_p$ in place of $\mathcal{Y}$ and $\mathcal{D}_p$ in place of $\mathcal{D}$, we find that there is some $m_* \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \|\Phi^t(U^0) - \Phi_m^t(U^0)\|_{\mathcal{Y}_p} < \delta/2$$

and

$$\sup_{t\in[0,T]} \|\Phi_m^t(U^0)\|_{\mathcal{Y}_{p+1}} \leq C$$

for some $C > 0$, all $m \geq m_*$, and all U^0 satisfying (3.11). Thus, with $\tilde{\delta} = \delta/2$ and

$$\tilde{\mathcal{D}}_{p+1} = \mathcal{D}_p \cap \operatorname{int} \mathcal{B}_{C+\delta}^{\mathcal{Y}_{p+1}}(0)\,,$$

where $\operatorname{int}(\mathcal{U})$ denotes the interior of a set $\mathcal{U}$ of a Banach space $\mathcal{X}$, condition (3.14) is satisfied for all $m \geq m_*$. This completes the proof. □

By combining this theorem with Theorem 2.4 we obtain convergence of the space-time discretization to the semiflow $\Phi^t(U^0)$ of order $O(h^p)+O(m^{-K-1})$ for $t \in [0,T]$ and $m \geq m_*$ with uniform bounds for all U^0 satisfying (3.11). In particular, we do not require a coupling between spatial resolution m and temporal resolution h for this convergence result.

3.3. **Accuracy of derivatives of Galerkin truncated time discretization.** Results corresponding to Theorems 3.2 and 2.8 hold true for the stability under spectral truncation of the numerical stage vector and its derivatives.

Theorem 3.6 (Projection error for the numerical method, local version)**.** *Assume* (A), (B1), (RK1), *and* (RK2). *Fix $R \in (0, \delta_*]$ such that $\mathcal{D}_K^{-R}$ is nonempty and choose $U^0 \in \mathcal{D}_K^{-R}$. Let $h_* = h_*(R, U^0) > 0$ and $m_* = m_*(R, U^0)$ be as in Theorem 3.2. Then for every $0 \leq P \leq K$,*

$$\|W - W_m\|_{N-1-P,K-P} = O(m^{-P}) \tag{3.15a}$$

and

$$\|\Psi^h - \Psi^h_m\|_{N-1-P,K-P} = O(m^{-P})\,, \tag{3.15b}$$

where the norm in (3.15) *is defined with respect to the spaces* (2.28). *The order constants depend only on the bounds afforded by* (B1) *and* (3.5), *on the coefficients of the method, and on* R.

Proof. The proof of (3.15) is an application of Theorem A.9 on the stability of contraction mappings where, as in the setting of Theorem 3.2, Π is defined by (3.2), $\Pi_m = \mathbb{P}_m \circ \Pi$, and we set

$$\mathcal{W}_j = \mathcal{B}^{\mathcal{Z}_j}_R(\mathbb{1}U^0) \qquad \text{where} \qquad \mathcal{Z}_j = \mathcal{Y}^s_j \tag{3.16}$$

for $j = 0, \dots, K$. We further identify $\mathcal{I} = (0, h_*)$, $w = W$, $\mu = h$, and $\mathcal{X} = \mathcal{Y}_K$ with $\mathcal{U} \equiv \mathcal{U}_K = \mathcal{B}^{\mathcal{Y}_K}_{R_*}(U^0)$.

We already verified in the proof of Theorem 3.2 that conditions (i) and (ii) of Theorem A.9 hold with uniform bounds for $m \geq m_*$. Thus, Theorem A.9 (b) yields a bound of the form (3.15a) provided we can show that

$$\|\!|\Pi - \Pi_m|\!\|_{N-1,K,0,P} = O(m^{-P})\,. \tag{3.17}$$

Writing

$$\Pi(W;U,h) - \Pi_m(W;U,h) = (\mathrm{id} - h\mathsf{a}A)^{-1}\,\mathbb{1}\mathbb{Q}_m U + h\mathsf{a}\,(\mathrm{id} - h\mathsf{a}A)^{-1}\,\mathbb{Q}_m B(W)\,, \tag{3.18}$$

we apply Lemma A.6 to both terms on the right as follows. For the first term,

$$G_m(W;U,h) \equiv \mathbb{Q}_m(\mathrm{id} - h\mathsf{a}A)^{-1}\mathbb{1}\,\mathbb{Q}_m U\,,$$

we take $\Pi = \mathrm{id}$, $\Sigma = \mathbb{P}_m$, and $v(U,h) = w(U,h) = (\mathrm{id} - h\mathsf{a}A)^{-1}\mathbb{1}U$ to conclude that there exists c_1 such that

$$\|\!|G_m|\!\|_{N-1,K,0,P} = \|G_m\|_{N-P-1,K-P} \leq c_1 \|\!|\mathbb{Q}_m|\!\|_{N-1,K,0,P} = O(m^{-P})\,, \tag{3.19}$$

where the first equality is due to (2.31), we recall Lemma 3.1 for the differentiability properties of $(\mathrm{id} - h\mathsf{a}A)^{-1}$, and note that the final statement is due to estimate (2.11) on the Galerkin remainder.

To estimate the second term on the right of (3.18), we apply Lemma A.6 for each $\kappa \in P, \dots, K$ with $\Pi = \mathrm{id}$, $\Sigma = \mathbb{P}_m$, $u = W$, and $v(u,h) = w(u,h) = h\mathsf{a}\,(\mathrm{id} - h\mathsf{a}A)^{-1}B(W)$, $\mathcal{U}$ replaced by $\mathcal{U}_\kappa \equiv \mathcal{B}^{\mathcal{Z}_\kappa}_R(\mathbb{1}U^0)$, N replaced by $N-1$, and K replaced by κ. Hence, by Lemma 3.1, there is some c_2 such that

$$\|\mathbb{Q}_m\, h\mathsf{a}\,(\mathrm{id} - h\mathsf{a}A)^{-1}B(W)\|_{N-1-P,\kappa-P} \leq c_2\, \|\!|\mathbb{Q}_m|\!\|_{N-1,\kappa,0,P} = O(m^{-P})\,.$$

Then, Lemma A.5 with N replaced by $N-1$ and $S = P$ implies

$$\|\!|\mathbb{Q}_m\, h\mathsf{a}\,(\mathrm{id} - h\mathsf{a}A)^{-1}B|\!\|_{N-1,K,0,P} = O(m^{-P})\,. \tag{3.20}$$

The constants c_1 and c_2 depend only on the bounds from (B1) and the bounds from Lemma 3.1. Altogether, we have proved (3.17); the proof of (3.15a) is complete.

To prove estimate (3.15b), note that by (3.3),

$$\Psi^h(U) - \Psi^h_m(U) = \mathsf{S}(hA)\mathbb{Q}_m U + \mathsf{b}^T\,(J(W(U,h),h) - \mathbb{P}_m J(W_m(U,h),h)\,, \tag{3.21}$$

where

$$J(W;U,h) = h(\mathrm{id} - h\mathsf{a}A)^{-1}B(W)\,, \tag{3.22}$$

so that

$$\|\Psi^h - \Psi^h_m\|_{N-1-P,K-P} \le \|\mathsf{S}(hA)\mathbb{Q}_m U\|_{N-1-P,K-P} + s\,\|\mathsf{b}\|\,\|J \circ W - \mathbb{P}_m J \circ W_m\|_{N-1-P,K-P}\,. \tag{3.23}$$

We estimate the first term of (3.23) using Lemma A.6 with $\Pi = \mathrm{id}$, $\Sigma = \mathbb{P}_m$, $u = U$, $\mathcal{Z}_j = \mathcal{Y}_j$, $\mathcal{W}_j = \mathcal{B}^{\mathcal{Y}_j}_R(U^0)$ for $j = 0, \dots, K$, and $w(u,h) = v(u,h) = \mathsf{S}(hA)U$. For the second term of (3.23), we use Lemma A.6 with $\Pi = J$, $\Sigma = \mathbb{P}_m J$, $\mathcal{Z}_j$ and $\mathcal{W}_j$ from (3.16) as before, $w(U,h) = W(U,h)$, and $v(U,h) = W_m(U,h)$. Thus, by Lemma 3.1, there exists c_4 such that

$$\|\Psi^h - \Psi^h_m\|_{N-1-P,K-P} \le c_4\,\|\mathbb{Q}_m U\|_{N-1,K,0,P} + c_4\,\|\mathbb{Q}_m J\|_{N-1,K,0,P} + c_4\,\|W - W_m\|_{N-1-P,K-P}\,. \tag{3.24}$$

The first term is $O(m^{-P})$ by (2.11); to obtain the required estimate for the second term we proceed as in the computation proving (3.20), but with $h(\mathrm{id} - h\mathsf{a}A)^{-1}B$ in place of $h\mathsf{a}(\mathrm{id} - h\mathsf{a}A)^{-1}B$; the third term is $O(m^{-P})$ by (3.15a). □

Theorem 3.7 (Projection error for the numerical method, uniform version)**.** *Assume* (A)*,* (B1)*,* (RK1)*, and* (RK2)*. Choose $\delta \in (0, \delta_*]$ small enough such that $\mathcal{D}^{-\delta}_{K+1}$ is nonempty, and let $h_* = h_*(\delta) > 0$ and $m_* = m_*(\delta)$ be as in Theorem 3.3. Then* (3.15) *holds true with respect to the uniform setting* (2.29)*. Moreover, for every $0 \le P \le K+1$ and $N > K+1$,*

$$\|W - W_m\|_{N-1-P,K+1-P} = O(m^{-P}) \tag{3.25a}$$

and, for a *invertible,*

$$\|\Psi^h - \Psi^h_m\|_{N-1-P,K+1-P} = O(m^{-P})\,, \tag{3.25b}$$

where the norm in (3.25) *is defined with respect to the spaces* (2.29)*. The order constants depend only on the bounds afforded by* (B1)*,* (2.16)*, and* (3.5)*, on the coefficients of the method, and on δ.*

Proof. For each $U^0 \in \mathcal{D}^{-\delta}_{K+1}$, define Π, Π_m, $\mathcal{W}_j = \mathcal{B}^{\mathcal{Y}^s_j}_{\tilde{R}}(\mathbb{1}U^0)$, $\mathcal{Z}_j = \mathcal{Y}^s_j$, $\mathcal{I} = (0, h_*)$, etc., as in the proof of Theorem 3.2. We first show that (3.15) holds true with respect to the uniform setting (2.29). We already verified in the proof of Theorem 3.3 that conditions (i) and (ii) of Theorem A.9 hold uniformly in $U^0 \in \mathcal{D}^{-\delta}_{K+1}$ and $m \ge m_*$. Further, by checking uniformity of all required estimates, we verify that (3.17) in the proof of Theorem 3.6 hold uniformly in $U^0 \in \mathcal{D}^{-\delta}_{K+1}$. Applying Theorem A.9 (b) for all $U^0 \in \mathcal{D}^{-\delta}_{K+1}$ then implies that (3.15a) holds in the uniform setting (2.29). Similarly, (3.24) holds uniformly in $U^0 \in \mathcal{D}^{-\delta}_{K+1}$ so that (3.15b) holds with respect to the uniform setting (2.29).

We next show that, for $N > K$,

$$\|AW - AW_m\|_{N-1-P,K-P} = O(m^{-P}) \tag{3.26a}$$

and, for a invertible,

$$\|A\Psi^h - A\Psi^h_m\|_{N-1-P,K-P} = O(m^{-P})\,, \tag{3.26b}$$

in the uniform setting (2.29). We apply A onto (3.18) as well as onto the expression for $\Psi^h - \Psi^h_m$. The resulting difference expressions are then estimated as in the proof of Theorem 3.6 using Lemma 3.1, in particular (3.5c). Now, we aim to show that, for $N > K+1$ and $0 \leq P \leq K$,

$$\|\partial_h W - \partial_h W_m\|_{N-P-2,K-P} = O(m^{-P}) \tag{3.27a}$$

and

$$\|\partial_h \Psi - \partial_h \Psi_m\|_{N-P-2,K-P} = O(m^{-P})\,. \tag{3.27b}$$

To prove (3.27a), we apply Theorem A.9 on the stability of contraction mappings to the pair Π from (3.2) and $\Pi_m = \mathbb{P}_m \circ \Pi$, but this time on the $(K+1)$ scale $\mathcal{Z}_j = \mathcal{Y}_j^s$ for $j = 0, \ldots, K$ and $\mathcal{Z}_{K+1} = \mathcal{Y}_K^s$ with $\mathcal{W}_j = \mathcal{D}_j^s$ for $j = 0, \ldots, K$ and $\mathcal{W}_{K+1} = \mathcal{D}_K^s$. Set, as before, $\mathcal{U} = \mathcal{D}_{K+1}^{-\delta}$, $\mathcal{X} = \mathcal{Y}_{K+1}$, and $\mathcal{I} = (0, h_*)$. Due to (3.6c) and (B1), the map Π_m satisfies the assumptions of Theorem A.9 in this setting for $m \geq m_*$. We obtain $\|\!|\Pi - \Pi_m|\!\|_{N-1,K+1,0,P} = O(m^{-P})$ as in the proof of Theorem 3.6. By Theorem A.9 (b), this implies $\|W - W_m\|_{N-P-1,K+1-P} = O(m^{-P})$ with respect to the above defined hierarchy, and in particular (3.27a).

Estimate (3.27b) is proved similarly. We estimate the norms of both terms in (3.21), with J as in (3.22). First, as in the proof of Theorem 3.6, using Lemma A.6 with $\Pi = \mathrm{id}$, $\Sigma = \mathbb{P}_m$, $u = U$, $\mathcal{Z}_j = \mathcal{Y}_j$, $\mathcal{W}_j = \mathcal{B}_r^{\mathcal{Y}_j}(U^0)$ for $j = 0, \ldots, K+1$, where $r > 0$ is such that $\mathcal{D}_{K+1} \subset \mathcal{B}_r^{\mathcal{Y}_{K+1}}(0)$, and $w(u,h) = v(u,h) = \mathsf{S}(hA)U$, we obtain

$$\|\mathsf{S}(hA)\mathbb{Q}_m U\|_{N-P-1,K+1-P} = O(m^{-P}).$$

Thus, in particular,

$$\|\partial_h \mathsf{S}(hA)\mathbb{Q}_m U\|_{N-P-2,K-P} = O(m^{-P}) \tag{3.28}$$

holds with respect to the uniform setting (2.29b).

We now estimate the second term of (3.21). Consider the $(K+1)$-scale from above, i.e., $\mathcal{Z}_j = \mathcal{Y}_j^s$ for $j = 0, \ldots, K$, $\mathcal{Z}_{K+1} = \mathcal{Y}_K^s$, $\mathcal{W}_j = \mathcal{D}_j^s$ for $j = 0, \ldots, K$, $\mathcal{W}_{K+1} = \mathcal{D}_K^s$, $\mathcal{U} = \mathcal{D}_{K+1}^{-\delta}$, $\mathcal{X} = \mathcal{Y}_{K+1}$, and $\mathcal{I} = (0, h_*)$. Due to (3.6c) and (B1), the maps J and $\mathbb{P}_m J$ satisfy the assumptions of Lemma A.6 in this setting, and, as we have seen above, W and W_m also satisfy the conditions of Lemma A.6 for this choice of scale and $m \geq m_*$. We obtain $\|\!|Q_m J|\!\|_{N-1,K+1,0,P} = O(m^{-P})$ as in the proof of Theorem 3.6, and

$$\|J \circ W - \mathbb{P}_m J \circ W_m\|_{N-1-P,K+1-P} = O(m^{-P})$$

for the above choice of scale. This implies that, with respect to the uniform setting (2.29b), we have

$$\|\partial_h (J \circ W - \mathbb{P}_m J \circ W_m)\|_{N-2-P,K-P} = O(m^{-P})$$

which, together with (3.28) and (3.21), implies (3.27b).

Finally, Lemma A.4, given that (3.15) holds in the uniform setting (2.29b) and together with estimates (3.26) and (3.27), implies (3.25). □

Remark 3.8. If, in the setting of Theorem 3.7, the matrix a is not assumed to be invertible, then (3.27b) still holds, cf. Remark 3.4.

Appendix A. Stability of contraction mappings

Abstract contraction mapping theorems on a scale of Banach spaces have been obtained in [15, 19, 21]. For the results in this paper, we must, in addition, estimate the stability of the fixed point under perturbation of the contraction map.

For $K \in \mathbb{N}_0$, let $\mathcal{Z} = \mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \ldots \supset \mathcal{Z}_K$ be a scale of Banach spaces, each continuously embedded in its predecessor, and let $\mathcal{V}_j, \mathcal{W}_j \subset \mathcal{Z}_j$ be nested sequences of sets. Let $\mathcal{X}$ be a Banach space, and let $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{I} \subset \mathbb{R}$ be open. We note that all results in this section easily extend to the case where $\mathcal{I}$ is an open subset of $\mathbb{R}^p$. We may assume that $\|w\|_{\mathcal{Z}_j} \leq \|w\|_{\mathcal{Z}_{j+1}}$ for all $w \in \mathcal{Z}_{j+1}$. (If this is not the case, we inductively equip $\mathcal{Z}_{j+1}$ with the equivalent norm $\|\cdot\|_{\mathcal{Z}_{j+1}} + \|\cdot\|_{\mathcal{Z}_j}$.)

As detailed in Section 2.3, we use the following additional integer indices. The minimal regularity we guarantee for the image space of the function considered is scale rung L, the "loss index" S indicates how many rungs on the scale the range of a function is down relative to its domain, and N denotes the maximal regularity of the function. We assume $0 \leq L \leq K - S \leq N - S$. We work with the family of spaces

$$\mathcal{C}_{N,K,L,S}(\{\mathcal{V}_j\},\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) = \bigcap_{\substack{i+j+k \leq N-S \\ L+\ell \leq k \leq K-S}} \mathcal{C}_{\mathrm{b}}^{(\underline{i},\underline{j},\ell)}(\mathcal{V}_{k+S} \times \mathcal{U} \times \mathcal{I}; \mathcal{W}_{k-\ell}), \tag{A.1a}$$

endowed with norm (2.30a), and abbreviate

$$\mathcal{C}_{N,K,L}(\{\mathcal{V}_j\},\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) = \mathcal{C}_{N,K,L,0}(\{\mathcal{V}_j\},\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}), \tag{A.1b}$$

$$\mathcal{C}_{N,K}(\{\mathcal{V}_j\},\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) = \mathcal{C}_{N,K,0,0}(\{\mathcal{V}_j\},\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) \tag{A.1c}$$

with corresponding norms (2.30b) and (2.30c), respectively. We note that any function of class (A.1a) has a maximal number of $N - L - S$ derivatives in its first and second argument on the lowest admissible domain scale $\mathcal{Z}_{L+S}$.

Furthermore, let

$$\mathcal{C}_{N,K,L}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) = \bigcap_{\substack{j+k \leq N \\ L+\ell \leq k \leq K}} \mathcal{C}_{\mathrm{b}}^{(\underline{j},\ell)}(\mathcal{U} \times \mathcal{I}; \mathcal{W}_{k-\ell}), \tag{A.1d}$$

endowed with norm (2.30d), and abbreviate

$$\mathcal{C}_{N,K}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) = \mathcal{C}_{N,K,0}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) \tag{A.1e}$$

with corresponding norm (2.30e). For future reference, we note the following.

Remark A.1. When a map $\Pi \in \mathcal{C}_{N,K,L,S}(\{\mathcal{V}_j\},\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})$ does not depend on w, it can be interpreted as an element from $\mathcal{C}_{N,K,L}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})$ where, by (2.31), $\|\!|\Pi|\!\|_{N,K,L,S} = \|\Pi\|_{N-S,K-S,L}$.

We simply write $\mathcal{C}_{N,K,L,S}$ and $\mathcal{C}_{N,K,L}$ when the arguments are unambiguous. We also write

$$\partial_\mu \Pi(w(u,\mu);u,\mu) = \partial_\mu \Pi(w;u,\mu)\big|_{w=w(u,\mu)} = (\partial_\mu \Pi \circ w)(u,\mu)$$

to denote partial μ-derivatives vs. $\mathrm{D}_\mu(\Pi(w(u,\mu),u,\mu))$ to denote full μ-derivatives.

We begin with four technical lemmas which can be proved by simple index arithmetic. Details can be found in [15].

Lemma A.2. *If $N > K$ then, with $\mathcal{W} \equiv \mathcal{W}_0$,*

$$\mathcal{C}_{N,K}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) \subset \mathcal{C}_{\mathrm{b}}^K(\mathcal{U} \times \mathcal{I};\mathcal{W}).$$

Lemma A.3. *Suppose that $w \in \mathcal{C}_{N,K,L}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})$ and that the map $(u,\tilde{u},\mu) \mapsto \mathrm{D}_u w(u,\mu)\tilde{u}$ is of class $\mathcal{C}_{N,K,L}(\mathcal{U}\times\mathcal{B}_1^{\mathcal{X}}(0),\mathcal{I};\{\mathcal{Z}_j\})$. Then $w \in \mathcal{C}_{N+1,K,L}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})$ and*

$$\|w\|_{N+1,K,L} \leq \sup_{\|\tilde{u}\|_{\mathcal{X}}\leq 1} \|\mathrm{D}_u w\,\tilde{u}\|_{N,K,L} + \|w\|_{N,K,L} \,.$$

Lemma A.4. *When $N > K$, $w \in \mathcal{C}_{N,K+1,L+1}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) \cap \mathcal{C}_{N,L,L}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})$, and $\partial_\mu w \in \mathcal{C}_{N-1,K,L}(\mathcal{U},\mathcal{I};\{\mathcal{Z}_j\})$, then*

$$w \in \mathcal{C}_{N,K+1,L}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})$$

and

$$\|w\|_{N,K+1,L} \leq \|w\|_{N,K+1,L+1} + \|w\|_{N,L,L} + \|\partial_\mu w\|_{N-1,K,L} \,.$$

Lemma A.5. *We have*

$$\bigcap_{S\leq\kappa\leq K} \mathcal{C}_{N-S,\kappa-S,L}(\mathcal{V}_\kappa\times\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) = \mathcal{C}_{N,K,L,S}(\{\mathcal{V}_j\},\mathcal{U},\mathcal{I};\{\mathcal{W}_j\}) \,,$$

and

$$|\!|\!|\Pi|\!|\!|_{\mathcal{C}_{N,K,L,S}(\{\mathcal{V}_j\},\mathcal{U};\mathcal{I};\{\mathcal{W}_j\})} \sim \max_{S\leq\kappa\leq K} \|\Pi\|_{\mathcal{C}_{N-S,\kappa-S,L}(\mathcal{V}_\kappa\times\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})} \,,$$

where $\sim$ denotes that left hand and right hand sides provide equivalent norms on $\mathcal{C}_{N,K,L,S}$.

We now prove a stability result for fixed points of contraction mappings, i.e., we want to bound norms of differences of fixed points in terms of norms of differences of contraction maps. To do so, we first need to look at a corresponding stability result for compositions.

The following lemma states that the difference between two functions which are both compositions of functions can be estimated by the difference of the outer functions and the difference of the inner functions, and that the same holds for derivatives of the difference. Here S is the minimal smoothness of the image of the inner functions and of the domain of the outer functions, K is the number of scales and $N-S$ is the maximal number of derivatives of the inner functions. Finally, P is used to relax the required smoothness in the estimates.

Lemma A.6 (Stability of compositions)**.** *For $0 \leq S+P \leq K \leq N$, suppose $\Pi = \Pi(w;u,\mu)$, $\Sigma = \Sigma(w;u,\mu)$, $w = w(u,\mu)$, and $v = v(u,\mu)$ satisfy*

(i) $\Pi, \Sigma \in \mathcal{C}_{N+1,K,0,S}(\{\mathcal{W}_j\},\mathcal{U},\mathcal{I};\{\mathcal{Z}_j\})$;
(ii) $w, v \in \mathcal{C}_{N,K,S}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})$.

Then:

(a) $\Pi\circ w \in \mathcal{C}_{N-S,K-S}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})$ *and* $\|\Pi\circ w\|_{N-S,K-S,L}$ *can be bounded by a polynomial with non-negative coefficients in* $|\!|\!|\Pi|\!|\!|_{N,K,L,S}$ *and* $\|w\|_{N,K,S+L}$*; the same holds true for* $\Sigma\circ v$.
(b) *There is some $c>0$ which is a polynomial with non-negative coefficients in* $|\!|\!|\Pi|\!|\!|_{N+1,K,0,S}$, $\|w\|_{N,K,S}$, $|\!|\!|\Sigma|\!|\!|_{N+1,K,0,S}$, *and* $\|v\|_{N,K,S}$ *such that*

$$\|\Pi\circ w - \Sigma\circ v\|_{N-P-S,K-P-S} \leq c\,|\!|\!|\Pi-\Sigma|\!|\!|_{N,K,0,P+S} + c\,\|w-v\|_{N-P,K-P,S} \,. \tag{A.2}$$

Remark A.7. Part (a) was already shown in [15, Lemma A.6] and is the chain rule on the scale of Banach spaces. It will be our main tool for obtaining estimates on the scale of Banach spaces for compositions of maps of the form $(\Pi\circ w)(u,\mu) \equiv \Pi(w(u,\mu);u,\mu)$. The essence of the result is very natural: When the outer function

Π loses S rungs on the scale, the inner function w must have minimal regularity $L = S$ and the composition maps at best into scale rung $K - S$.

Proof. We prove part (b) only. It follows the same pattern as the proof of part (a) with the additional difficulty that we need to carefully keep track of differences in the various spaces. We proceed by induction in N and K. For $N = K = P + S$,

$$\begin{aligned}\|\Pi \circ w - \Sigma \circ v\|_{\mathcal{C}(\mathcal{U}\times\mathcal{I};\mathcal{Z})} &\leq \|\Pi \circ w - \Pi \circ v\|_{\mathcal{C}(\mathcal{U}\times\mathcal{I};\mathcal{Z})} + \|\Pi \circ v - \Sigma \circ v\|_{\mathcal{C}(\mathcal{U}\times\mathcal{I};\mathcal{Z})} \\ &\leq c_0 \, \|w - v\|_{\mathcal{C}(\mathcal{U}\times\mathcal{I};\mathcal{Z}_S)} + |\!|\!|\Pi - \Sigma|\!|\!|_{P+S,P+S,0,P+S} \,,\end{aligned}$$

where, by the mean value theorem, $c_0 = |\!|\!|\Pi|\!|\!|_{P+S+1,P+S,0,P+S}$.

Let us now increment N holding P and K fixed. Let $\mathcal{B} \equiv \mathcal{B}_1^{\mathcal{X}}(0)$. By Lemma A.3, it is sufficient to derive the claimed upper bound for the $\mathcal{C}_{N-P-S,K-P-S}(\mathcal{U} \times \mathcal{B}, \mathcal{I}; \{\mathcal{W}_j\})$ norm of the function which maps $((u, \tilde u), \mu) \in (\mathcal{U} \times \mathcal{B}) \times \mathcal{I}$ to

$$\begin{aligned}\mathrm{D}_u(\Pi \circ w - \Sigma \circ v)\, \tilde u &= (\partial_u \Pi \, \tilde u) \circ w - (\partial_u \Sigma \, \tilde u) \circ v \\ &\quad + \hat\Pi \circ (\mathrm{D}_u w \, \tilde u) - \hat\Sigma \circ (\mathrm{D}_u v \, \tilde u) \,, \end{aligned} \tag{A.3}$$

where $\hat\Pi$ and $\hat\Sigma$ are defined in (A.8) below.

To estimate the first line of the right of (A.3), we define $\Pi_1(w; (u, \tilde u), \mu) = \partial_u \Pi(w; u, \mu)\, \tilde u$ and $\Sigma_1(w; (u, \tilde u), \mu) = \partial_u \Sigma(w; u, \mu)\, \tilde u$. Then, by induction hypothesis, there is a constant c_1 which is a polynomial in $|\!|\!|\Pi|\!|\!|_{N+1,K,0,S} \geq |\!|\!|\Pi_1|\!|\!|_{N,K,0,S}$, $|\!|\!|\Sigma|\!|\!|_{N+1,K,0,S}$, $\|w\|_{N+1,K,S} \geq \|w\|_{N,K,S}$, and $\|v\|_{N+1,K,S}$ such that

$$\begin{aligned}&\|\Pi_1 \circ w - \Sigma_1 \circ v\|_{N-P-S,K-P-S} \\ &\quad \leq c_1 \, |\!|\!|\Pi_1 - \Sigma_1|\!|\!|_{N,K,0,P+S} + c_1 \, \|w - v\|_{N-P,K-P,S} \\ &\quad \leq c_1 \, |\!|\!|\Pi - \Sigma|\!|\!|_{N+1,K,0,P+S} + c_1 \, \|w - v\|_{N-P,K-P,S} \,.\end{aligned}$$

To estimate the second line of the right of (A.3), fix

$$r = \max\{\|w\|_{N+1,K,S}, \|v\|_{N+1,K,S}\} \tag{A.4}$$

and set $\mathcal{V}_j = \mathcal{B}_r^{\mathcal{Z}_j}(0)$ for $j = S, \ldots, K$. By Lemma A.8 (a) below, the maps $\hat\Pi$, $\hat\Sigma$ satisfy condition (i), i.e.,

$$\hat\Pi, \hat\Sigma \in \mathcal{C}_{N+1,K,0,S}(\{\mathcal{V}_j\}, \mathcal{U}, \mathcal{I}; \{\mathcal{Z}_j\}) \,.$$

For fixed $\tilde u \in \mathcal{B}$, a direct estimate verifies that $\hat w((u, \tilde u), \mu) = \mathrm{D}_u w(u, \mu)\, \tilde u$ and $\hat v((u, \tilde u), \mu) = \mathrm{D}_u v(u, \mu)\, \tilde u$ map $(\mathcal{U} \times \mathcal{B}) \times \mathcal{I}$ into each of the domains $\mathcal{V}_S, \ldots, \mathcal{V}_K$ of $\hat\Pi$ and $\hat\Sigma$. Hence, $\hat w$ and $\hat v$ satisfy assumption (ii). Then, by induction hypothesis, there is some constant c_2 such that

$$\begin{aligned}&\|\hat\Pi \circ \hat w - \hat\Sigma \circ \hat w\|_{N-P-S,K-P-S} \\ &\quad \leq c_2 \, |\!|\!|\hat\Pi - \hat\Sigma|\!|\!|_{N,K,0,P+S} + c_2 \, \|\hat w - \hat v\|_{N-P,K-P,S} \\ &\quad \leq c_2 \, |\!|\!|\hat\Pi - \hat\Sigma|\!|\!|_{N,K,0,P+S} + c_2 \, \|w - v\|_{N+1-P,K-P,S} \,.\end{aligned} \tag{A.5}$$

We note that c_2 is a polynomial in $\|w\|_{N,K,S}$, $\|v\|_{N,K,S}$, $|\!|\!|\hat\Pi|\!|\!|_{N+1,K,0,S}$ which, by Lemma A.8 (a), is bounded by a polynomial in $\|w\|_{N+1,K,S}$, $|\!|\!|\Pi|\!|\!|_{N+2,K,0,S}$, and r, and $|\!|\!|\hat\Sigma|\!|\!|_{N+1,K,0,S}$ which is bounded by a polynomial in $\|v\|_{N+1,K,S}$, $|\!|\!|\Sigma|\!|\!|_{N+2,K,0,S}$, and r.

To estimate the term $\|\hat{\Pi} - \hat{\Sigma}\|_{N,K,0,P+S}$ in the last inequality above, note that the maps

$$\Pi_2(w; (\hat{w}, u), \mu) = \mathrm{D}_w \Pi(w; u, \mu)\, \hat{w} \quad \text{and} \quad \Sigma_2(w; (\hat{w}, u), \mu) = \mathrm{D}_w \Sigma(w; u, \mu)\, \hat{w}$$

satisfy, for $P + S \le \kappa \le K$,

$$\Pi_2, \Sigma_2 \in \mathcal{C}_{N+1,\kappa,0,S}(\{\mathcal{W}_j\}, \mathcal{V}_\kappa \times \mathcal{U}, \mathcal{I}; \{\mathcal{Z}_j\})\,.$$

So the induction hypothesis applies once again, asserting that there is a constant c_3 such that

$$\|\Pi_2 \circ w - \Sigma_2 \circ v\|_{N-P-S,\kappa-P-S} \le c_3\, \|\Pi_2 - \Sigma_2\|_{N,\kappa,0,P+S} + c_3\, \|w - v\|_{N-P,\kappa-P,S}$$

where, for all $P + S \le \kappa \le K$,

$$\|\Pi_2 - \Sigma_2\|_{N,\kappa,0,P+S} \le r\, \|\Pi - \Sigma\|_{N+1,K,0,P+S}$$

and c_3 is a polynomial in $\|\Pi\|_{N+2,K,0,S}$, $\|\Sigma\|_{N+2,K,0,S}$, $\|w\|_{N+1,K}$, $\|v\|_{N+1,K}$, and r. Therefore, by Lemma A.5, there is some constant c_4, which is independent of Π_2, Σ_2, v, and w, such that

$$\begin{aligned}
\|\hat{\Pi} - \hat{\Sigma}\|_{N,K,0,P+S} &= \|\Pi_2 \circ w - \Sigma_2 \circ v\|_{N,K,0,P+S} \\
&\le c_4 \max_{P+S \le \kappa \le K} \|\Pi_2 \circ w - \Sigma_2 \circ v\|_{N-P-S,\kappa-P-S} \\
&\le r\, c_3\, c_4\, \|\Pi - \Sigma\|_{N+1,K,0,P+S} + c_3\, c_4\, \|w - v\|_{N-P,K-P,S}\,.
\end{aligned}$$

This last assertion is summarized in Lemma A.8 (with N replaced by $N - 1$). This concludes the inductive step in N.

Second, we prove that the conclusion also holds when we increment $K - S$ when $K < N$, holding N fixed. By Lemma A.4,

$$\begin{aligned}
\|\Pi \circ w - \Sigma \circ v\|_{N-P-S,K-P-S+1} &\le \|\Pi \circ w - \Sigma \circ v\|_{N-P-S,K-P-S+1,1} \\
&\quad + \|\Pi \circ w - \Sigma \circ v\|_{N-P-S,0} + \|\mathrm{D}_\mu(\Pi \circ w - \Sigma \circ v)\|_{N-P-S-1,K-P-S}\,. \qquad \text{(A.6)}
\end{aligned}$$

To estimate the first term on the right, note that we can apply the induction hypothesis on the translated scale $\tilde{\mathcal{Z}}_j = \mathcal{Z}_{j+1}$, $\tilde{\mathcal{W}}_j = \mathcal{W}_{j+1}$. Thus, there is a constant c_5 with the required polynomial dependence such that

$$\begin{aligned}
&\|\Pi \circ w - \Sigma \circ v\|_{\mathcal{C}_{N-P-S,K-P-S+1,1}(\mathcal{U},\mathcal{I};\{\mathcal{Z}_j\})} \\
&\quad = \|\Pi \circ w - \Sigma \circ v\|_{\mathcal{C}_{N-P-S,K-P-S}(\mathcal{U},\mathcal{I};\{\tilde{\mathcal{Z}}_j\})} \\
&\quad \le c_5\, \|\Pi - \Sigma\|_{\mathcal{C}_{N,K,0,P+S}(\{\tilde{\mathcal{W}}_j\},\mathcal{U},\mathcal{I};\{\tilde{\mathcal{Z}}_j\})} + c_5\, \|w - v\|_{\mathcal{C}_{N-P,K-P,S}(\mathcal{U},\mathcal{I};\{\tilde{\mathcal{Z}}_j\})} \\
&\quad = c_5\, \|\Pi - \Sigma\|_{\mathcal{C}_{N,K+1,1,P+S}(\{\mathcal{W}_j\},\mathcal{U},\mathcal{I};\{\mathcal{Z}_j\})} + c_5\, \|w - v\|_{\mathcal{C}_{N-P,K+1-P,1+S}(\mathcal{U},\mathcal{I};\{\mathcal{Z}_j\})}\,.
\end{aligned}$$

For the second term on the right of (A.6), we apply the induction hypothesis on the trivial scale, obtaining that there is a constant c_6 with the required polynomial dependence such that

$$\|\Pi \circ w - \Sigma \circ v\|_{N-P-S,0} \le c_6\, \|\Pi - \Sigma\|_{N,P+S,0,P+S} + c_6\, \|v - w\|_{N-P,S,S}\,.$$

For the third term on the right of (A.6), we estimate

$$\|\mathrm{D}_\mu(\Pi\circ w)-\mathrm{D}_\mu(\Sigma\circ v)\|_{N-P-S-1,K-P-S} \le \|\partial_\mu\Pi\circ w-\partial_\mu\Sigma\circ v\|_{N-P-S-1,K-P-S} + \|\hat\Pi\circ\partial_\mu w-\hat\Sigma\circ\partial_\mu v\|_{N-P-S-1,K-P-S}\,. \tag{A.7}$$

To estimate the first term on the right of (A.7), notice that $\Pi,\Sigma\in\mathcal{C}_{N+1,K+1,0,S}$ implies $\partial_\mu\Pi,\partial_\mu\Sigma\in\mathcal{C}_{N+1,K+1,0,S+1}$. Since $w,v\in\mathcal{C}_{N,K+1,1+S}$, we conclude that $\partial_\mu\Pi$, $\partial_\mu\Sigma$, w and v satisfy the assumptions of the lemma. Since $K-S$ is not incremented, the induction hypothesis applies and proves that there is a constant c_7 with the required polynomial dependence such that

$$\begin{aligned}
&\|\partial_\mu\Pi\circ w-\partial_\mu\Sigma\circ v\|_{N-P-(S+1),K+1-P-(S+1)}\\
&\quad\le c_7\,\|\!|\partial_\mu\Pi-\partial_\mu\Sigma|\!\|_{N,K+1,0,P+S+1}+c_7\,\|w-v\|_{N-P,K+1-P,S+1}\\
&\quad\le c_7\,\|\!|\Pi-\Sigma|\!\|_{N,K+1,0,P+S}+c_7\,\|w-v\|_{N-P,K+1-P,S}\,.
\end{aligned}$$

To estimate the second term on the right of (A.7), we fix

$$r=\max\{\|w\|_{N,K+1,S},\|v\|_{N,K+1,S}\}\,,$$

set $\mathcal{V}_j=\mathcal{B}_r^{\mathcal{Z}_j}(0)$, and recall from above that $\hat\Pi,\hat\Sigma\in\mathcal{C}_{N,K}(\{\mathcal{V}_j\},\mathcal{U},\mathcal{I};\{\mathcal{Z}_j\})$, cf. Lemma A.8. Then, $\partial_\mu w$ and $\partial_\mu v$ map $\mathcal{U}\times\mathcal{I}$ into each of the domains $\mathcal{V}_S,\dots,\mathcal{V}_K$ of $\hat\Pi$, $\hat\Sigma$. Applying the induction hypothesis to $\hat\Pi$, $\hat\Sigma$ and $\partial_\mu w$, $\partial_\mu v$, we obtain that there exists a constant c_8 such that

$$\begin{aligned}
&\|\hat\Pi\circ\partial_\mu w-\hat\Sigma\circ\partial_\mu v\|_{N-P-S-1,K-P-S}\\
&\qquad\le c_8\,\|\!|\hat\Pi-\hat\Sigma|\!\|_{N-1,K,0,P+S}+c_8\,\|\partial_\mu w-\partial_\mu v\|_{N-1-P,K-P,S}\\
&\qquad\le c_8\,\|\!|\hat\Pi-\hat\Sigma|\!\|_{N-1,K,0,P+S}+c_8\,\|w-v\|_{N-P,K+1-P,S}\,.
\end{aligned}$$

The first term on the right hand side is estimated as before, yielding a bound of the form (A.9).

We have thus found the required upper bounds for all terms on the right of (A.6), thereby completing the inductive step also when K is incremented. □

In the proof of Lemma A.6, we used part (a) and proved statement (b) of the following lemma which we state for later reference. A proof of part (a) can be found in [15, Lemma A.7].

Lemma A.8. *Let Π, Σ, w, and v be as in Lemma A.6; let $r>0$, $\mathcal{V}_j=\mathcal{B}_r^{\mathcal{Z}_j}(0)$ for $j=S,\dots,K$, $0\le P\le\min(N-1,K)$, and*

$$\hat\Pi(\hat w;u,\mu)=(\partial_w\Pi\circ w)(u,\mu)\,\hat w\quad\text{and}\quad\hat\Sigma(\hat w;u,\mu)=(\partial_w\Sigma\circ v)(u,\mu)\,\hat w \tag{A.8}$$

Then:

(a) *$\hat\Pi\in\mathcal{C}_{N-1,K,0,S}(\{\mathcal{V}_j\},\mathcal{U},\mathcal{I};\{\mathcal{Z}_j\})$ with $\|\!|\hat\Pi|\!\|_{N-1,K,0,S}$ bounded by a polynomial in $\|w\|_{N-1,K,S}$ and $r\,\|\!|\Pi|\!\|_{N,K,0,S}$, and the same holds true for $\hat\Sigma$.*

(b) *There is some polynomial $c\ge0$ in $\|\!|\Pi|\!\|_{N+1,K,0,S}$, $\|\!|\Sigma|\!\|_{N+1,K,0,S}$, $\|w\|_{N,K,S}$, $\|v\|_{N,K,S}$, and r such that*

$$\|\!|\hat\Pi-\hat\Sigma|\!\|_{N-1,K,0,P+S}\le c\,\|\!|\Pi-\Sigma|\!\|_{N,K,0,P+S}+c\,\|w-v\|_{N-1-P,K-P,S}\,. \tag{A.9}$$

Now we are ready to prove the result on the stability of fixed points of contraction mappings on scales of Banach spaces.

Theorem A.9 (Stability of contraction mappings). *For $N, K \in \mathbb{N}_0$ with $N \geq K$, let $\mathcal{Z} = \mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \ldots \supset \mathcal{Z}_K$ be a scale of Banach spaces, each continuously embedded in its predecessor, let $\mathcal{W}_j \subset \mathcal{Z}_j$ be a nested sequence of closures of open sets, let $\mathcal{X}$ be a Banach space, and let $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{I} \subset \mathbb{R}$ be open. Assume*

(i) $\Pi, \Sigma \in \mathcal{C}_{N+1,K}(\{\mathcal{W}_j\}, \mathcal{U}, \mathcal{I}; \{\mathcal{W}_j\})$;
(ii) *$w \mapsto \Pi(w; u, \mu)$ and $v \to \Sigma(v; u, \mu)$ are contractions on $\mathcal{W}_j$ with contraction constant $c'_j < 1$ uniformly for all $u \in \mathcal{U}$, $\mu \in \mathcal{I}$, and $j = 0, \ldots, K$.*

Then the following is true.

(a) *The fixed point equation $\Pi(w; u, \mu) = w$ has a unique solution*
$$w \in \mathcal{C}_{N+1,K}(\mathcal{U}, \mathcal{I}; \{\mathcal{W}_j\})$$
and $\|w\|_{N+1,K}$ is bounded by a function which is a polynomial with nonnegative coefficients in $\|\Pi\|_{N+1,K}$ and $(1 - c'_j)^{-1}$. The same holds true for the fixed point $v = \Sigma(v; u, \mu)$.
(b) *Let $P \leq K \leq N$. Then there is some polynomial c with nonnegative coefficients in $\|\Pi\|_{N+1,K}$, $\|\Sigma\|_{N+1,K}$, and $(1 - c'_j)^{-1}$ for $j = 0, \ldots, K$, such that*
$$\|w - v\|_{N-P,K-P} \leq c \, \|\Pi - \Sigma\|_{N,K,0,P} \, .$$

Part (a) is a version of a contraction mapping theorem on a scale of Banach spaces which was already proved in [15, Theorem A.9].

Proof. We prove part (b) only. It follows the same pattern as the proof of part (a) with the additional difficulty that we need to carefully keep track of differences in the various spaces. As before, we use induction in N and K. For $N = K = P$, we must estimate
$$\begin{aligned} \|w - v\|_{\mathcal{Z}} &\leq \|\Pi(w; u, \mu) - \Pi(v; u, \mu)\|_{\mathcal{Z}} + \|\Pi(v; u, \mu) - \Sigma(v; u, \mu)\|_{\mathcal{Z}} \\ &\leq c'_0 \|w - v\|_{\mathcal{Z}} + \|\Pi(v; u, \mu) - \Sigma(v; u, \mu)\|_{\mathcal{Z}} \, , \end{aligned}$$
where c'_0 is the common contraction parameter with respect to the $\mathcal{Z}_0$ norm. Therefore,
$$\|w - v\|_{\mathcal{C}(\mathcal{U} \times \mathcal{I}; \mathcal{Z})} \leq \frac{1}{1 - c'_0} \|\Pi - \Sigma\|_{\mathcal{C}(\mathcal{W}_P \times \mathcal{U} \times \mathcal{I}; \mathcal{Z})} = \frac{1}{1 - c'_0} \|\Pi - \Sigma\|_{P,P,0,P} \, .$$

We first prove that the conclusion also holds when we increment N, holding K fixed. By Lemma A.3,
$$\|w - v\|_{N+1-P,K-P} \leq \sup_{\|\tilde{u}\| \leq 1} \|(\mathrm{D}_u w - \mathrm{D}_u v) \, \tilde{u}\|_{N-P,K-P} + \|w - v\|_{N-P,K-P} \, . \tag{A.10}$$

By induction hypothesis, there is a constant c_1 which is a polynomial in $\|\Pi\|_{N+1,K}$, $\|\Sigma\|_{N+1,K}$, and $(1 - c'_j)^{-1}$ for $j = 0, \ldots, K$ such that
$$\|w - v\|_{N-P,K-P} \leq c_1 \, \|\Pi - \Sigma\|_{N,K,0,P} \, .$$
It remains to compute an appropriate bound for the first term on the right of (A.10).

Note that $\tilde{w}(\hat{u},\mu) = \mathrm{D}_u w(u,\mu)\,\tilde{u}$, where $\hat{u} = (u,\tilde{u}) \in \mathcal{U} \times \mathcal{B}$, $\mathcal{B} = \mathcal{B}_1^{\mathcal{X}}(0)$, is a fixed point of the contraction map $\tilde{\Pi}$ given by

$$\begin{aligned} \tilde{\Pi}(\tilde{w};(u,\tilde{u}),\mu) &= \partial_w \Pi(w(u,\mu);u,\mu)\,\tilde{w} + \partial_u \Pi(w(u,\mu);u,\mu)\,\tilde{u} \\ &\equiv \hat{\Pi}(\tilde{w};u,\mu) + \partial_u \Pi(w(u,\mu);u,\mu)\,\tilde{u}\,. \end{aligned} \tag{A.11}$$

Similarly, $\tilde{v}(\hat{u},\mu) = \mathrm{D}_u v(u,\mu)\,\tilde{u}$ is a fixed point of $\tilde{\Sigma}$. From part (a) we know that $w, v \in \mathcal{C}_{N+1,K}(\mathcal{U},\mathcal{I};\{\mathcal{W}_j\})$. Setting

$$r = \max\{|\!|\!|\Pi|\!|\!|_{N+1,K}, |\!|\!|\Sigma|\!|\!|_{N+1,K}\} \max_{j=0,\dots,K} \frac{1}{1-c'_j} \tag{A.12}$$

and $\mathcal{V}_j = \mathcal{B}_r^{\mathcal{Z}_j}(0)$ for $j = 0,\dots,K$, we find by Lemma A.6 (a) and Lemma A.8 (a) that $\tilde{\Pi}, \tilde{\Sigma} \in \mathcal{C}_{N+1,K}(\{\mathcal{V}_j\},\mathcal{U}\times\mathcal{B},\mathcal{I};\{\mathcal{V}_j\})$. Hence, $\tilde{\Pi}$ and $\tilde{\Sigma}$ satisfy the assumptions of the theorem and, by induction hypothesis, there is some constant c_2, depending polynomially on $(1-c'_j)^{-1}$ for $j = 0,\dots,K$, $|\!|\!|\tilde{\Pi}|\!|\!|_{N+1,K}$, and $|\!|\!|\tilde{\Sigma}|\!|\!|_{N+1,K}$ such that

$$\begin{aligned} \|\tilde{w}-\tilde{v}\|_{N-P,K-P} &\le c_2\, |\!|\!|\tilde{\Pi}-\tilde{\Sigma}|\!|\!|_{N,K,0,P} \\ &\le c_2\, |\!|\!|\hat{\Pi}-\hat{\Sigma}|\!|\!|_{N,K,0,P} + c_2\, \|(\partial_u \Pi \circ w - \partial_u \Sigma \circ v)\,\tilde{u}\|_{N-P,K-P} \end{aligned} \tag{A.13}$$

where, in the second inequality, we refer to definition (A.8) of $\hat{\Pi}$ and $\hat{\Sigma}$ and to Remark A.1. By Lemma A.6 (a) and Lemma A.8 (a), taking note of Remark A.1, the norms

$$|\!|\!|\tilde{\Pi}|\!|\!|_{N+1,K} \le |\!|\!|\hat{\Pi}|\!|\!|_{N+1,K} + \|(\partial_u \Pi \circ w)\,\tilde{u}\|_{N+1,K}$$

and $|\!|\!|\tilde{\Sigma}|\!|\!|_{N+1,K}$ are polynomials in $|\!|\!|\Pi|\!|\!|_{N+2,K}$, $|\!|\!|\Sigma|\!|\!|_{N+2,K}$, $\|w\|_{N+1,K}$, $\|v\|_{N+1,K}$, and r. Due to the definition of r in (A.12) and part (a), these quantities, hence the constants in (A.13), have bounds that can be chosen as polynomials in $|\!|\!|\Pi|\!|\!|_{N+2,K}$, $|\!|\!|\Sigma|\!|\!|_{N+2,K}$, and $(1-c'_j)^{-1}$ for $j = 0,\dots,K$.

Applying Lemma A.8 to the first term on the right hand side of the second line of (A.13) and Lemma A.6 to the second term, both with $S = 0$, we find that there is a constant c_3 depending polynomially on $|\!|\!|\Pi|\!|\!|_{N+2,K}$, $|\!|\!|\Sigma|\!|\!|_{N+2,K}$, $\|w\|_{N+1,K}$, and $\|w\|_{N+1,K}$ such that

$$\begin{aligned} \|\tilde{w}-\tilde{v}\|_{N-P,K-P} &\le c_3\, |\!|\!|\Pi-\Sigma|\!|\!|_{N+1,K,0,P} + c_3\, \|w-v\|_{N-P,K-P} \\ &\le c_3\, |\!|\!|\Pi-\Sigma|\!|\!|_{N+1,K,0,P} + c_4\, |\!|\!|\Pi-\Sigma|\!|\!|_{N,K,0,P} \\ &\le c_5\, |\!|\!|\Pi-\Sigma|\!|\!|_{N+1,K,0,P}\,. \end{aligned}$$

In the second inequality we have used the induction hypothesis so that c_4 and c_5 are polynomials in $|\!|\!|\Pi|\!|\!|_{N+2,K}$, $|\!|\!|\Sigma|\!|\!|_{N+2,K}$, and $(1-c'_j)^{-1}$ for $j = 0,\dots,K$. This concludes the inductive step in N.

Second, we prove that the conclusion also holds when we increment $K < N$, holding N fixed. Recall from Lemma A.4 that

$$\begin{aligned} \|w-v\|_{N-P,K-P+1} &\le \|w-v\|_{N-P,K-P+1,1} \\ &\quad + \|w-v\|_{N-P,0} + \|\partial_\mu w - \partial_\mu v\|_{N-P-1,K-P}\,; \end{aligned} \tag{A.14}$$

we will estimate the three norms on the right hand side separately. For the first norm note that a translation of the scale with $\tilde{\mathcal{Z}}_j = \mathcal{Z}_{j+1}$ and the induction hypothesis show that

$$\|w - v\|_{N-P,K-P+1,1} \leq c_6 \, |\!|\!|\Pi - \Sigma|\!|\!|_{N,K+1,1,P} \,, \tag{A.15}$$

where c_6 is a polynomial in $|\!|\!|\Pi|\!|\!|_{N+1,K+1} \geq |\!|\!|\Pi|\!|\!|_{N+1,K+1,1}$, $|\!|\!|\Sigma|\!|\!|_{N+1,K+1}$, and $(1 - c_j')^{-1}$ for $j = 0, \dots, K+1$.

For the second term on the right of (A.14), we apply the induction hypothesis on the trivial scale, so that is a constant c_7 such that

$$\|w - v\|_{N-P,0} \leq c_7 \, |\!|\!|\Pi - \Sigma|\!|\!|_{N,P,0,P} \,.$$

For the third term on the right of (A.14), we note that $\tilde{w} = \partial_\mu w$ and $\tilde{v} = \partial_\mu v$ are fixed points of the respective contraction maps $\tilde{\Pi}$ and $\tilde{\Sigma}$ of the form

$$\begin{aligned} \tilde{\Pi}(\tilde{w}; u, \mu) &= \partial_w \Pi(w(u,\mu); u, \mu) \, \tilde{w} + \partial_\mu \Pi(w(u,\mu); u, \mu) \\ &\equiv \hat{\Pi}(\tilde{w}; u, \mu) + \partial_\mu \Pi(w(u,\mu); u, \mu) \,. \end{aligned} \tag{A.16}$$

By part (a), $v, w \in \mathcal{C}_{N,K+1}(\mathcal{U}, \mathcal{I}; \{\mathcal{Z}_j\})$. Setting

$$r = \max\{|\!|\!|\Pi|\!|\!|_{N,K+1}, |\!|\!|\Sigma|\!|\!|_{N,K+1}\} \max_{j=0,\dots,K} \frac{1}{1 - c_j'}$$

and $\mathcal{V}_j = \mathcal{B}_r^{\mathcal{Z}_j}(0)$ for $j = 0, \dots, K$, we find that, by Lemma A.6 (a) and Lemma A.8 (a), $\tilde{\Pi}, \tilde{\Sigma} \in \mathcal{C}_{N,K}(\{\mathcal{V}_j\}, \mathcal{U}, \mathcal{I}; \{\mathcal{V}_j\})$. Hence, $\tilde{\Pi}$ and $\tilde{\Sigma}$ satisfy the assumptions of the theorem and, by induction hypothesis, there is some constant c_8, depending polynomially on $(1 - c_j')^{-1}$, $|\!|\!|\tilde{\Pi}|\!|\!|_{N,K}$, and $|\!|\!|\tilde{\Sigma}|\!|\!|_{N,K}$ such that

$$\begin{aligned} \|\tilde{w} - \tilde{v}\|_{N-P-1,K-P} &\leq c_8 \, |\!|\!|\tilde{\Pi} - \tilde{\Sigma}|\!|\!|_{N-1,K,0,P} \\ &\leq c_8 \, |\!|\!|\hat{\Pi} - \hat{\Sigma}|\!|\!|_{N-1,K,0,P} + c_8 \, \|\partial_\mu \Pi \circ w - \partial_\mu \Sigma \circ v\|_{N-P-1,K-P} \end{aligned} \tag{A.17}$$

where, in the second inequality, we refer to definition (A.8) of $\hat{\Pi}$ and $\hat{\Sigma}$ and to Remark A.1. By Lemma A.6 (a) and Lemma A.8 (a), taking note of Remark A.1, the norms

$$|\!|\!|\tilde{\Pi}|\!|\!|_{N,K} \leq |\!|\!|\hat{\Pi}|\!|\!|_{N,K} + \|\partial_\mu \Pi \circ w\|_{N,K}$$

and $|\!|\!|\tilde{\Sigma}|\!|\!|_{N,K}$ are polynomials in $|\!|\!|\Pi|\!|\!|_{N+1,K+1}$, $|\!|\!|\Sigma|\!|\!|_{N+1,K+1}$, $\|w\|_{N,K+1}$, $\|v\|_{N,K+1}$, and r. Due to the definition of r, these quantities, hence the constant c_8 in (A.17) has a bound that can be chosen as a polynomial in $|\!|\!|\Pi|\!|\!|_{N+1,K+1}$, $|\!|\!|\Sigma|\!|\!|_{N+1,K+1}$, and $(1 - c_j')^{-1}$ for $j = 0, \dots, K$.

The first term in the second line of (A.17) is estimated by Lemma A.8. We obtain

$$\begin{aligned} |\!|\!|\hat{\Pi} - \hat{\Sigma}|\!|\!|_{N-1,K,0,P} &\leq c_9 \, |\!|\!|\Pi - \Sigma|\!|\!|_{N,K,0,P} + c_9 \, \|w - v\|_{N-P,K-P} \\ &\leq c_{10} \, |\!|\!|\Pi - \Sigma|\!|\!|_{N,K,0,P} \,. \end{aligned} \tag{A.18}$$

Here c_9 is a polynomial in $|\!|\!|\Pi|\!|\!|_{N+1,K+1} \geq |\!|\!|\Pi|\!|\!|_{N,K}$, $|\!|\!|\Sigma|\!|\!|_{N+1,K+1}$, $\|w\|_{N,K+1} \geq \|w\|_{N-1,K}$ and $\|v\|_{N,K+1}$, and we have used the induction hypothesis in the last inequality, with c_{10} a polynomial in $|\!|\!|\Pi|\!|\!|_{N+1,K+1}$, $|\!|\!|\Sigma|\!|\!|_{N+1,K+1}$, and $(1 - c_j')^{-1}$ for $j = 0, \dots K$.

For the second term on the right of (A.17), note that the hypothesis of the theorem, with K replaced by $K+1$, implies that

$$\partial_\mu \Pi, \partial_\mu \Sigma \in \mathcal{C}_{N,K+1,0,1}(\{\mathcal{W}_j\}, \mathcal{U}, \mathcal{I}; \{\mathcal{Z}_j\}) ,$$

so that Lemma A.6 applied with $S=1$ yields a constant c_{11} which is a polynomial in $|||\Pi|||_{N,K+1} \geq |||\partial_\mu \Pi|||_{N,K+1,0,1}$, $|||\Sigma|||_{N,K+1}$, $\|w\|_{N,K+1} \geq \|w\|_{N,K+1,1}$, and $\|v\|_{N,K+1}$ such that

$$\begin{aligned} \|(\partial_\mu \Pi) \circ w - (\partial_\mu \Sigma) \circ v\|_{N-P-1,K-P} & \\ \leq c_{11} \, |||\partial_\mu \Pi - \partial_\mu \Sigma|||_{N,K+1,0,P+1} + c_{11} \, \|w - v\|_{N-P,K+1-P,1} & \\ \leq c_{11} \, |||\Pi - \Sigma|||_{N,K+1,0,P} + c_{11} \, c_{12} \, |||\Pi - \Sigma|||_{N,K+1,1,P} & \\ \leq c_{13} \, |||\Pi - \Sigma|||_{N,K+1,0,P} \, , & \qquad \text{(A.19)} \end{aligned}$$

where the second term in the third inequality is due to (A.15), and c_{12} and c_{13} depend polynomially on the required quantities. Inserting (A.18) and (A.19) into (A.17) then concludes the inductive step in K. □

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