110231 Nonlinear Dynamics Lab

Exercise Sheet

'Simple model of a neuron'

Here we will explore a well-known minimal model of a biological neuron, the FitzHugh-Nagumo (FHN) oscillator. Despite containing only two dynamical variables, a voltage-like variable v and a recovery variable w, this system exhibits a wide range of behaviors, e.g. oscillations, excitability and bistability. The main task of the first part of this segment (Exercises 1–3) is to understand, which parameter values give rise to these behaviors. In the second part (Exercises 4–6) we couple several FHN oscillators and study the resulting spatiotemporal dynamics.
We will work with the following version of the FHN equations:

$$\begin{aligned} \frac{dv}{dt} &= [(a-v)(v-1)v - w]\,\frac{1}{e} \\ \frac{dw}{dt} &= bv - \gamma w + c\,. \end{aligned} \tag{1}$$

Please use *Mathematica* for the following set of exercises.

Exercise 1: First numerical exploration. Plot the nullclines of the FHN system for parameter values a=-1, b=0.12, c=0, e=1 and γ=0.12. Investigate the effect of varying the parameters b and c on the nullclines. Describe this qualitatively. Simulate the trajectories of v and w with time for the first set of parameter values. Now simulate them for the same parameter values, but for e=0.1. What can you say about the effect of this change? Explain the dynamical role of the parameter e.

Exercise 2: Excitability. Plot the nullclines of the FHN system for parameter values a=-1, b=0.12, c=0.03, e=1 and γ=0.12. Simulate v and w over time, allow the system to settle into a steady state and then perturb the system from its fixed point by increasing the value of v by first a "small" amount (e.g. 0.01) and then by a "large" amount (e.g. 0.3). How does this modify the trajectories of the dynamical variables v and w? Display this as plots of v and w against time and in the corresponding phase plane plots. Describe both sets of plots qualitatively.
What features of the system do you think would make it suitable as a mathematical model of biological (in particular, neuronal) dynamics?

Exercise 3: Linear stability analysis. Here we analyze the behavior of the system as a function of the parameters c and b.

- Parameter c: Keep the other parameters as in Exercise 1 and compute the fixed points and their stability (via the eigenvalues of the Jacobian matrix!) as a function of c (varying c between -0.2 and 0.2). Compare the predictions of dynamical behavior derived from the eigenvalues at the fixed points with the trajectories you simulated.

- Parameter b: Keep the other parameters as in Exercise 1 and compute the fixed points and their stability (via the eigenvalues of the Jacobian matrix!) as a function of b (varying b between 0.1 and 0.6). Compare the predictions of dynamical behavior derived from the eigenvalues at the fixed points with the trajectories you simulated. Does the system display bistable behavior? In which parameter range?

Exercise 4: Chain of coupled FHN oscillators. Use the following form of (linear) coupling:

$$\begin{aligned}\frac{dv_i}{dt} &= [(a - v_i)(v_i - 1)v_i - w_i]\,\frac{1}{e} + D\,[v_{i+1} + v_{i-1} - 2v_i] \\ \frac{dw_i}{dt} &= bv_i - \gamma w_i + c \end{aligned} \tag{2}$$

where D is the diffusion constant (use values for D between 0.1 and 0.5). Simulate this system of coupled ODEs and plot a space-time plot. Use random initial conditions. Then use the steady state as initial conditions, except for one oscillator in the center of the chain, where you perturb v as in Exercise 2 by an additive shift of 0.3. Observe and interpret the behavior.

Exercise 5: Spiral waves in a simple model of excitable dynamics. In order to further explore spatiotemporal patterns generated by excitable dynamics, we turn to a simple time-discrete model operating on a discrete state space Σ consisting of 'excited' E, 'susceptible' S and 'refractory' R. A susceptible element goes into the excited state, when it has an excited neighbor, $S \xrightarrow{E \in \mathrm{NB}} E$, where NB stands for the neighborhood of an element; an excited element always enters the refractory state, $E \xrightarrow{1} R$; a refractory element has a probability p to enter the susceptible state, $R \xrightarrow{p} S$. Additionally, with a probability f an element can go from the susceptible state to the excited state, $S \xrightarrow{f} E$. For the following exercise we consider the deterministic limit[1] ($p = 1$, $f = 0$). Implement this minimal model and run it on a 20×20 lattice, starting from the following initial conditions: All lattice sites are in the susceptible state S, except for a horizontal line of ten elements with coordinates $(10, 1), (10, 2), ..., (10, 10)$, which are set to E, and a horizontal line of ten elements with coordinates $(9, 1), (9, 2), ..., (9, 10)$, which are set to R. Simulate the resulting spatiotemporal pattern. Explain, which features of the initial conditions are relevant for observing this behavior.

Exercise 6 (Bonus exercise): Spiral waves in coupled FHN oscillators. Repeat Exercise 5 with coupled FHN oscillators. Guiding questions: How does the coupling in Eq. (2) generalize to the 2D lattice case? How can the initial conditions of the model from Exercise 5 be implemented in the FHN system?

[1] We will revisit the general model near the end of the semester to study such excitable dynamics on random graphs.