The discrete and fast Fourier transforms

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1 Fourier series

We begin by recalling the familiar definition of the Fourier series. For a periodic function $u\colon [0, 2\pi] \to \mathbb{C}$, we define the *Fourier transform*

$$\hat{u}_k = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kx} \, u(x) \, \mathrm{d}x \, . \tag{1}$$

for every $k \in \mathbb{Z}$. By direct computation, noting that the boundary terms of the integral cancel due to periodicity, we obtain the *orthogonality relation*

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}jx} \, \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x = \delta_{ij} \, , \tag{2}$$

where δ_{jk} denotes the *Kronecker symbol*

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \, , \\ 0 & \text{otherwise} \, . \end{cases} \tag{3}$$

We conclude that the *inverse Fourier transform* is given by

$$u(x) = \sum_{k \in \mathbb{Z}} \hat{u}_k \, \mathrm{e}^{\mathrm{i}kx} \, , \tag{4}$$

which can be verified by inserting (4) into (1), changing the order of integration and summation, and applying the orthogonality relation (2). In the same way, we obtain the *Parseval identity*

$$\frac{1}{2\pi} \int_0^{2\pi} \overline{u(x)} \, v(x) \, \mathrm{d}x = \sum_{k \in \mathbb{Z}} \overline{\hat{u}_k} \, \hat{v}_k \, , \tag{5}$$

where the overbar denotes the complex conjugate.

Remark 1. The choice of the interval of definition of the periodic function u is arbitrary, but choosing $[0, 2\pi[$ has notational advantages. Other intervals can be obtained by translation and scaling. The prefactor on the right of (1) is sometimes fully or partially moved into the inverse Fourier transform (4).

Remark 2. More elegantly, we may speak of functions on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The right hand notation means that two real numbers are considered equivalent if they differ by an integer multiple of 2π.

Remark 3. The precise functional setting is not important for our purposes, and we will not dwell on this issue. Let us remark that this theory is developed by noting that (a) the inversion formula (4) is a continuous linear map from the sequence space ℓ^1 into the space of essentially bounded functions $L^\infty(\mathbb{T})$ and that (b) the Parseval identity (5) shows that the Fourier transform is an isometry with respect to the norms of the larger respective spaces ℓ^2 and $L^2(\mathbb{T})$, and can thus be extended to those larger spaces using a density argument. The details depend on advanced Analysis and may be found, for example, in [2].

Remark 4. The orthogonality relation (2) expresses that $\{\mathrm{e}^{\mathrm{i}kx} \colon k \in \mathbb{Z}\}$ is an orthonormal set with respect to the inner product

$$\langle u, v \rangle = \frac{1}{2\pi} \int_0^{2\pi} \overline{u(x)}\, v(x)\, \mathrm{d}x \,. \tag{6}$$

It is a nontrivial result that this set is a *Hilbert basis* of the function space $L^2(\mathbb{T})$, which means that every $u \in L^2(\mathbb{T})$ can be expressed as a countable linear combination of basis functions [2]. From this point of view, the inverse Fourier transform (4) is nothing but the representation of a function in terms of a Hilbert basis.

2 The discrete Fourier transform

Let $v_0, \ldots, v_{N-1}$ be an N-tuple of real or complex numbers. We define the *discrete Fourier transform* (DFT) to be the linear map

$$\tilde{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}kjh}\, v_j \tag{7}$$

where $k = 0, \ldots, N-1$ and

$$h = \frac{2\pi}{N} \,. \tag{8}$$

Remark 5. We can think of $v_0, \ldots, v_{N-1}$ as the samples of a 2π-periodic function v on the equidistantly spaced nodes

$$x_j = jh \,. \tag{9}$$

In this case, we can write the formula for the DFT as

$$\tilde{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}kx_j}\, v(x_j) \,, \tag{10}$$

which looks like a Riemann sum approximation to the continuous Fourier transform (1). This interpretation, however, has some subtleties which will be discussed further in Section 3.

We now investigate the inverse of the discrete Fourier transform (7). To this end, we first derive the *discrete orthogonality relation*

$$
\begin{aligned}
\frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}kjh}\,\mathrm{e}^{\mathrm{i}ljh} = \frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{\mathrm{i}(l-k)jh} &= \frac{1}{N}\sum_{j=0}^{N-1} \Bigl(\mathrm{e}^{\mathrm{i}2\pi\frac{l-k}{N}}\Bigr)^{j} \\
&= \begin{cases} 1 & \text{if } l = k + mN \text{ for some } m \in \mathbb{Z} \\ \dfrac{1-q^N}{1-q} = 0 & \text{otherwise} \end{cases} \\
&= \delta_{kl}^{\mathrm{per}} \,,
\end{aligned} \tag{11}
$$

where

$$
q = \mathrm{e}^{\mathrm{i}2\pi\frac{l-k}{N}} \tag{12}
$$

so that $q^N = \mathrm{e}^{2\pi\mathrm{i}(l-k)} = 1$, and where $\delta_{kl}^{\mathrm{per}}$ denotes the *periodic Kronecker symbol*

$$
\delta_{kl}^{\mathrm{per}} = \begin{cases} 1 & \text{if } l = k + mN \text{ for some } m \in \mathbb{Z}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{13}
$$

Remark 6. Along the same lines, it is easy to show that for any $m \in \mathbb{Z}$,

$$
\frac{1}{N}\sum_{j=m+1}^{m+N} \mathrm{e}^{-\mathrm{i}kjh}\,\mathrm{e}^{\mathrm{i}ljh} = \delta_{kl}^{\mathrm{per}}\,. \tag{14}
$$

Remark 7. In the language of linear algebra, (11) states that the family of vectors $\{\boldsymbol{e}_k = N^{-1/2}\,(\mathrm{e}^{\mathrm{i}kx_0}, \dots, \mathrm{e}^{\mathrm{i}kx_{N-1}})\colon k = 0, \dots, N-1\}$ form an orthonormal basis of $\mathbb{C}^N$ with respect to the standard inner product.

Exercise 1. Prove the *discrete Parseval identity*

$$
\sum_{k=0}^{N-1} |\tilde{v}_k|^2 = \frac{1}{N}\sum_{j=0}^{N-1} |v_j|^2\,. \tag{15}
$$

Exercise 2. Let $\tilde{v}_k$ be defined by (7) for any $k \in \mathbb{Z}$. Show that $\tilde{v}_{k+N} = \tilde{v}_k$. Vice versa, set $v_{j+N} = v_j$, then show that for every $m \in \mathbb{Z}$,

$$
\tilde{v}_k = \frac{1}{N}\sum_{j=m+1}^{m+N} \mathrm{e}^{-\mathrm{i}kx_j}\,v_j\,. \tag{16}
$$

Exercise 3. Let

$$
w_j = \frac{1}{N}\sum_{l=0}^{N-1} u_l\,v_{j-l} \tag{17}
$$

with the understanding that the three N-tuples are periodically extended beyond their basic range of definition on which the index varies from 0 to $N-1$; see Exercise 2. Show that

$$
\tilde{w}_k = \tilde{u}_k\,\tilde{v}_k\,. \tag{18}
$$

Remark 8. Exercise 2 shows, in particular, that our choice of range for k, the "wavenumber" index, is arbitrary. Any range of N consecutive wavenumbers will do.

We conclude that the *inverse discrete Fourier transform* is given by

$$v_j = \sum_{k=0}^{N-1} \tilde{v}_k \, \mathrm{e}^{\mathrm{i}kx_j} \,, \tag{19}$$

as can be directly verified by substituting the discrete Fourier transform (7) into the right hand sum and applying the orthogonality relation (11).

Remark 9. The normalization convention is arbitrary. Many software implementations of the discrete Fourier transform, including those in Matlab and Octave, have the factor $1/N$ in the inverse transform (19) rather than in the forward transform (7). The convention above has the advantage that the DFT can be seen as the Riemann sum approximation of the continuous Fourier transform. If both the forward and the inverse transform get a factor of $1/\sqrt{N}$, the transform is unitary, which has certain theoretical advantages, but is usually avoided in actual code.

3 Sampling and approximation

From the linear algebra perspective, the previous section already gives a complete description of the discrete Fourier transform. In practice, however, the DFT is often used as an approximation to the continuous Fourier transform. Thus, we need to define the reconstruction of a function on $[0, 2\pi]$ and must then discuss the quality of the approximation.

To begin, assume that $u \colon [0, 2\pi] \to \mathbb{C}$ is continuous and periodic, and write $u_j = u(x_j)$ for $j = 0, \ldots, N-1$ to denote the values of u sampled on the grid. For simplicity, we assume that N is even. We now define a function v, the *trigonometric interpolant* of u on the grid, by letting $\tilde{v}_k$ be the DFT of the samples u_j,

$$\tilde{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}kx_j} \, u_j \tag{20}$$

for $k = -N/2, \ldots, N/2 - 1$, then set

$$v(x) = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k \, \mathrm{e}^{\mathrm{i}kx} \,. \tag{21}$$

Note that k still ranges over a set of N consecutive wave numbers. The particular choice here seems odd, but will turn out to be crucial for keeping the approximation error small.

The name "trigonometric interpolant" is justified by the fact that v and u coincide on the grid, to be verified in the following straightforward exercise.

Exercise 4. Show that $v(x_j) = u_j$ for $j = 0, \ldots, N-1$ and that $\hat{v}_k = \tilde{u}_k$ for $k = -N/2, \ldots, N/2-1$.

However, the discrete Fourier coefficients $\hat{v}_k = \tilde{u}_k$ are generally different from the Fourier coefficients $\hat{u}_k$ defined by (1)! To quantify how they differ, we compute, using (20), (4), and (11), that

$$\tilde{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}kx_j} \sum_{l \in \mathbb{Z}} \hat{u}_l \, \mathrm{e}^{\mathrm{i}lx_j} = \sum_{l \in \mathbb{Z}} \hat{u}_l \, \delta_{kl}^{\mathrm{per}} \,. \tag{22}$$

Moving the term where $k = l$ onto the left side of the equality and re-indexing the sum, we obtain the *Poisson summation formula* or *aliasing formula*

$$\tilde{u}_k - \hat{u}_k = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \hat{u}_{k+mN} \,. \tag{23}$$

This result shows that the error in the Fourier coefficients with wavenumbers within the range $k = -N/2, \ldots, N/2-1$ depends on the magnitude of $\hat{u}_l$ outside of this range. Since smooth functions have Fourier coefficients whose magnitude decreases with increasing $|k|$ (a more concrete version of this statement is given below), and real-valued functions have $|\hat{u}_k| = |\hat{u}_{-k}|$, the maximally symmetric choice of wavenumber range where $k = -N/2, \ldots, N/2-1$ can be expected to minimize the approximation error for "typical" functions.

Remark 10. The aliasing formula implies a version of the *Shannon sampling theorem*, often stated as "Exact reconstruction of a continuous-time baseband signal from its samples is possible if the signal is bandlimited and the sampling frequency is greater than twice the signal bandwidth" [6].

It is often useful to quantify the approximation error $u - v$ directly. It is easiest to do under the assumption that

$$\|u\|_{H^m}^2 = \sum_{k \in \mathbb{Z}} (1 + |k|^2)^m \, |\hat{u}_k|^2 < \infty \,. \tag{24}$$

Remark 11. The condition above is a statement about the smoothness of u. More specifically, it says that derivatives of u, while they may not exist everywhere, do not diverge too rapidly. For $m = 1$, for example, (24) is equivalent to saying that

$$\int_0^{2\pi} \bigl(|u(x)|^2 + |u'(x)|^2 \bigr) \, \mathrm{d}x < \infty \,. \tag{25}$$

Remark 12. The set of L^2 functions, i.e. square integrable functions, for which (24) holds, is called the Sobolev space H^m.

We compute

$$
\begin{aligned}
\|u-v\|_{L^2}^2 &= \int_0^{2\pi} |u-v|^2 \,\mathrm{d}x = 2\pi \sum_{k\in\mathbb{Z}} |\hat{u}_k - \hat{v}_k|^2 \\
&= 2\pi \sum_{k=-N/2}^{N/2-1} |\hat{u}_k - \hat{v}_k|^2 + 2\pi \left(\sum_{k<-N/2} + \sum_{k\geq N/2} \right) |\hat{u}_k|^2 \\
&= 2\pi \sum_{k=-N/2}^{N/2-1} \left| \sum_{j\neq 0} \hat{u}_{k+jN} \right|^2 + 2\pi \left(\sum_{k<-N/2} + \sum_{k\geq N/2} \right) |\hat{u}_k|^2 \\
&\leq 2\pi \left(\sum_{|k|\geq N/2} |\hat{u}_k| \right)^2 + 2\pi \sum_{|k|\geq N/2} |\hat{u}_k|^2 \,,
\end{aligned} \tag{26}
$$

where the last inequality is based on $\sum a_i^2 \leq \left(\sum a_i\right)^2$ for nonnegative a_i, and where we have symmetrized the range of the summation index taking the more conservative bound. The second term on the right of (26) is the *truncation error*, which can be estimated as

$$
\begin{aligned}
\sum_{|k|\geq N/2} |\hat{u}_k|^2 &= \sum_{|k|\geq N/2} |k|^{-2m} \, |k|^{2m} \, |\hat{u}_k|^2 \\
&\leq \left(\frac{N}{2} \right)^{-2m} \sum_{|k|\geq N/2} |k|^{2m} \, |\hat{u}_k|^2 \\
&= c_1(m) \, N^{-2m} \, \|u\|_{H^m}^2 \,.
\end{aligned} \tag{27}
$$

The first term in (26) is the *aliasing error*, which is estimated as

$$
\begin{aligned}
\left(\sum_{|k|\geq N/2} |\hat{u}_k| \right)^2 &= \left(\sum_{|k|\geq N/2} |k|^{-m} \, |k|^m \, |\hat{u}_k| \right)^2 \\
&\leq \sum_{|k|\geq N/2} |k|^{-2m} \sum_{|k|\geq N/2} |k|^{2m} \, |\hat{u}_k|^2 \\
&\leq c_2(m) \, N^{-2m+1} \, \|u\|_{H^m}^2 \,,
\end{aligned} \tag{28}
$$

where the first inequality is due to the Cauchy–Schwarz inequality, and the second inequality is a result of estimating the sum in terms of an integral,

$$
\sum_{|k|\geq \kappa} |k|^{-2m} \leq 2 \int_{\kappa-1}^{\infty} k^{-2m} \,\mathrm{d}k = \frac{2}{2m-1} \, (\kappa-1)^{1-2m} \,. \tag{29}
$$

Inserting (27) and (28) into (26), we have proved the following.

Theorem 1. *For every $m > \frac{1}{2}$ there exists a constant $c(m)$ such that for every $u \in H^m$*

$$
\|u-v\|_{L^2} \leq c(m) \, N^{1/2-m} \, \|u\|_{H^m} \,, \tag{30}
$$

where v is the trigonometric interpolant of u as given by (21).

4 The fast Fourier transform

The fast Fourier transform is mathematically equivalent to the discrete Fourier transform, but organizes the operations in a tree-like structure which reduces to the computational complexity from the $O(N^2)$ when viewed as a generic vector-matrix multiplication to $O(N \ln N)$ provided that N is completely factorizable into small prime factors.

The construction can be understood as follows. Assume that N can be written as a product of integers $N = pq$. Now re-index the discrete Fourier transform, writing

$$j = q\, j_1 + j_2 \quad \text{with } j_1 = 0, \dots, p-1 \text{ and } j_2 = 0, \dots, q-1\,, \tag{31}$$

$$k = p\, k_2 + k_1 \quad \text{with } k_1 = 0, \dots, p-1 \text{ and } k_2 = 0, \dots, q-1\,. \tag{32}$$

The discrete Fourier transform (7) then reads

$$\begin{aligned}
\tilde{v}_k &= \frac{1}{N} \sum_{j_2=0}^{q-1} \sum_{j_1=0}^{p-1} \mathrm{e}^{-\mathrm{i}(qj_1+j_2)(pk_2+k_1)h}\, v_j \\
&= \frac{1}{N} \sum_{j_2=0}^{q-1} \mathrm{e}^{-\mathrm{i}j_2(pk_2+k_1)h} \sum_{j_1=0}^{p-1} \mathrm{e}^{-\mathrm{i}j_1 k_2 pqh}\, \mathrm{e}^{-\mathrm{i}k_1 j_1 qh}\, v_j \\
&= \frac{1}{N} \sum_{j_2=0}^{q-1} \mathrm{e}^{-\mathrm{i}k_2 j_2 2\pi/q}\, \mathrm{e}^{-\mathrm{i}k_1 j_2 h} \sum_{j_1=0}^{p-1} \mathrm{e}^{-\mathrm{i}k_1 j_1 2\pi/p}\, v_j\,.
\end{aligned} \tag{33}$$

The crucial observation is that the inner sum is now itself, for j_2 fixed, a discrete Fourier transform of length p. Thus, if T_p denotes the number of terms needed to compute a length p Fourier transform, the computation of the inner sum for all values of j_2 requires $q\,T_p$ terms.

Similarly, the outer sum is, for k_1 fixed, a discrete Fourier transform of length q acting on the values of the inner sum pre-multiplied by $\exp(-\mathrm{i}k_1 j_2 h)$. Thus, its computation involves $p\,T_q$ terms.

We conclude that, in total,

$$T_N = q\,T_p + p\,T_q \tag{34}$$

which, in general, is less than $N^2 = p^2 q^2$. Moreover, the factorization of the DFT is recursive so long as p or q can themselves be factored.

In the important special case where $N = 2^k$, it is easy to solve this recurrence. Write $t_k = T_N$, $q = 2$ and $p = 2^{k-1}$, whence (34) takes the form

$$t_k = 2\,t_{k-1} + 2^{k-1}\,t_1\,. \tag{35}$$

Noting that self-consistency of this expression requires that $t_0 = 0$ (computing a length-one discrete Fourier transform is trivial and does not require any floating point operations), we can easily find the solution to this linear difference equation [4],

$$t_k = k\,2^{k-1}\,t_1\,. \tag{36}$$

Thus,

$$T_N = \log_2 N \, 2^{\log_2 N - 1} \, t_1 = N \log_2 N \, t_1/2 = O(N \ln N) \,. \tag{37}$$

More generally, if $N = p_1 \ldots p_m$, then the discrete Fourier transform can be computed in $N(p_1 + \ldots p_m)$ operations [1].

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