General Mathematics and Computational Science I

Difference Equations

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Abstract

These notes provide a first introduction to the qualitative theory of difference equations. We introduce the concept of equilibrium points, periodic points, and stability to derive a general description of the dynamics of a solution to a difference equation even in situations where a complete closed-form solution does not exist. These concepts are illustrated by several examples from modeling real-world systems. To a large part, this material is distilled from the book by Elaydi [1], which contains a wealth of more advanced topics and further examples.

1 Introduction

In general, a *difference equation* of *order* k is a relation of the form

$$x_{n+1} = f(x_n, \dots, x_{n-k+1}) \tag{1}$$

together with k starting values $x_0, \dots, x_{k-1}$. A sequence of real numbers x_n with $n = 0, 1, \dots$ is called the *solution* to the difference equation if it satisfies (1) as well as the given starting values. The set $\{x_n \colon n = 0, 1, \dots\}$ is called an *orbit*.

A difference equation is called *linear* if it is of the form

$$x_{n+1} = a_0(n)\, x_n + a_1(n)\, x_{n-1} + \dots + a_{k-1}(n)\, x_{n-k+1} + b(n)\,. \tag{2}$$

If the a_i do not depend on n, we speak of a *constant coefficient* difference equation; if $b(n) = 0$ we say that the equation is *homogeneous*.

We have already encountered difference equations before. For example, the Fibonacci sequence

$$a_{n+1} = a_n + a_{n-1}\,, \tag{3a}$$

$$a_0 = a_1 = 1\,, \tag{3b}$$

is a linear, second-order, constant coefficient, homogeneous difference equation. We already know that such equations can be solved by using the method of generating functions.

The main point here is to develop techniques that remain applicable when closed-form solutions are not available. However, we will frequently use solvable equations as examples. A relatively simple class of such equations is first order linear difference equations, which are discussed next.

2 First order linear difference equations

Problem 1. To treat a chronic disease, a dose of d mg of a drug is administered starting from day. Each day, a fraction of p of the drug is eliminated from the body. What is the amount of drug in the body after a very long time?

Let a_n denote the amount of drug in the body at day n. Suppose the treatment starts at day $n = 1$. Then $a_0 = 0$ and

$$a_{n+1} = (1 - p)\, a_n + d\,. \tag{4}$$

This is a first order linear difference equation which can be solved as follows. On day n, the dose which was administered on day k has been in the body for $n - k$ days. Of this dose, the amount remaining after each day is $1 - p$ times the amount that was present the previous day. So the amount left of the dose at day n is $(1 - p)^{n-k}\, d$. Thus,

$$\begin{aligned} a_n &= \sum_{k=1}^{n} (1-p)^{n-k}\, d \\ &= d \sum_{j=0}^{n-1} (1-p)^j \\ &= d\, \frac{1 - (1-p)^n}{p}\,, \end{aligned} \tag{5}$$

where, in the last step, we used the formula for the geometric series which we encountered before. In particular, as n becomes large, a_n converges to the *equilibrium* amount of drug in the body

$$\lim_{n\to\infty} a_n = \frac{d}{p}\,. \tag{6}$$

The same reasoning can be applied to general first order linear difference equations with nonconstant coefficients,

$$x_{n+1} = a_n\, x_n + b_n\,, \tag{7a}$$

$$x_0 = c\,. \tag{7b}$$

By tracing out the same steps as before, we obtain the following general form of the solution.

Theorem 1. *The solution to the general first order difference equation* (7) *is given by*

$$x_n = c \prod_{j=0}^{n-1} a_j + \sum_{k=0}^{n-1} b_k \prod_{j=k+1}^{n-1} a_j\,. \tag{8}$$

Remark 1. Note that the sum and product in (8) are nested, i.e. the expression reads

$$\sum_{k=0}^{n-1} b_k \prod_{j=k+1}^{n-1} a_j = b_0\, a_1 \cdots a_{n-1} + \cdots + b_{n-2}\, a_{n-1} + b_{n-1}\,. \tag{9}$$

Proof. We prove this formula by induction. For $n = 0$, the formula clearly reproduces the initial value $x_0 = c$. Let us now assume that the formula holds for some $n \geq 0$. Then

$$\begin{aligned} x_{n+1} &= a_n\, x_n + b_n \\ &= a_n \left(c \prod_{j=0}^{n-1} a_j + \sum_{k=0}^{n-1} b_k \prod_{j=k+1}^{n-1} a_j \right) + b_n \\ &= c \prod_{j=0}^{n} a_j + \sum_{k=0}^{n-1} b_k \prod_{j=k+1}^{n} a_j + b_n \\ &= c \prod_{j=0}^{n} a_j + \sum_{k=0}^{n} b_k \prod_{j=k+1}^{n} a_j\,. \end{aligned} \tag{10}$$

This completes the inductive step. □

Example 1. The linear, constant coefficient equation

$$x_{n+1} = a\, x_n + b\,, \tag{11}$$

an instance of which we encountered already in Problem 1, has solution

$$\begin{aligned} x_n &= x_0 \prod_{j=0}^{n-1} a + \sum_{k=0}^{n-1} b \prod_{j=k+1}^{n-1} a \\ &= a^n\, x_0 + b \sum_{k=0}^{n-1} a^{n-1-k} \\ &= a^n\, x_0 + b \sum_{j=0}^{n-1} a^j \\ &= \begin{cases} x_0 + n\, b & \text{if } a = 1 \\ a^n\, x_0 + b\, \dfrac{a^n - 1}{a - 1} & \text{if } a \neq 1\,, \end{cases} \end{aligned} \tag{12}$$

where the first step is a direct use of the general expression (8) and the last step is again due to the formula for the geometric series.

Example 2. The difference equation

$$x_{n+1} = (n+1)\, x_n + 2^n\, (n+1)!\,, \tag{13}$$

$$x_0 = 1 \tag{14}$$

has solution

$$
\begin{aligned}
x_n &= \prod_{j=0}^{n-1}(j+1) + \sum_{k=0}^{n-1} 2^k\,(k+1)!\,\prod_{j=k+1}^{n-1}(j+1) \\
&= n! + \sum_{k=0}^{n-1} 2^k\,(k+1)!\,\frac{n!}{(k+1)!} \\
&= n! + n!\,\frac{2^n-1}{2-1} \\
&= n!\,2^n\,.
\end{aligned}
\tag{15}
$$

3 Stability of equilibrium points

It is rare for a nonlinear, even first order, difference equation to have a closed-form solution. We thus seek means to qualitatively describe properties of the solution that can be deduced from the equation itself independent of whether a closed-form solution is available. Often, the index n plays the role of a time-like quantity in the modeling regime and the difference equation describes how a dependent quantity changes in time. In this context, difference equations are often referred to as *discrete dynamical systems*. The adjective "discrete" refers the time-like quantity being an integer index, not a real number.

This section introduces the important concept of equilibrium points and discusses their stability. We motivate the theory using an (explicitly solvable) example from economics.

Problem 2. A commodity is freely traded. The suppliers of this commodity increase production whenever the price is high and decrease production when the price is low. Buyers, on the other hand, react adversely to changes in price. Moreover, due to the time it takes a supplier to complete a unit of the product, changes in production always lag behind changes in price. Under the hypothesis that the market price is the price at which supply equals demand, does the market adjust to a stable equilibrium?

Note that this problem is not fully specified yet, and that we also have not yet defined the precise meaning of "stable" and "equilibrium". To proceed further, we make the simplest modeling assumptions that capture the essence of this problem.

First, we assume discrete market periods, labeled with integer n. Let p_n denote the price of the commodity, d_n the demand, and s_n the supply in period n. Second, we assume that the price-demand and price-supply relationships are linear, i.e.

$$d_n = -m_{\mathrm{d}}\,p_n + b_{\mathrm{d}} \tag{16}$$

and

$$s_{n+1} = m_{\mathrm{s}}\,p_n + b_{\mathrm{s}}\,, \tag{17}$$

where m_{d} and m_{s} are positive. The delay in the reaction of the supply side to changes in price is manifest in (17) through the dependence of the supply on the price of the *previous* market period. Finally, the market price hypothesis is expressed as

$$d_n = s_n \,. \tag{18}$$

By combining (16), (17), and (18), we obtain a first order linear difference equation for p_n, namely

$$p_{n+1} = -\frac{m_{\mathrm{s}}}{m_{\mathrm{d}}}\, p_n + \frac{b_{\mathrm{d}} - b_{\mathrm{s}}}{m_{\mathrm{d}}} \,. \tag{19}$$

An *equilibrium price* p^* is a price that does not change from one period to the next. Thus, at the equilibrium, we must have

$$p^* = -\frac{m_{\mathrm{s}}}{m_{\mathrm{d}}}\, p^* + \frac{b_{\mathrm{d}} - b_{\mathrm{s}}}{m_{\mathrm{d}}} \tag{20}$$

and therefore

$$p^* = \frac{b_{\mathrm{d}} - b_{\mathrm{s}}}{m_{\mathrm{d}} - m_{\mathrm{s}}} \,. \tag{21}$$

The behavior of this model can be visualized in a so-called *cobweb diagram*. The difference equation (19) is of the form $p_{n+1} = f(p_n)$. The procedure is as follows.

- Draw the graph of f into an x-y coordinate plane.
- Choose an initial value p_0 on the x-axis.
- Find the first iterate by moving vertically until intersecting the graph of f.
- Find the next iterate by moving horizontally until hitting the line $x = y$, then vertically until intersecting the graph of f. Repeat.

Figures 1–3 show the resulting cobweb diagrams for three different values of the ratio $m_{\mathrm{s}}/m_{\mathrm{d}}$. We can distinguish three regimes.

(i) When $m_{\mathrm{s}}/m_{\mathrm{d}} < 1$, the slope of the graph of f has magnitude less than one. The distance from the equilibrium price p^* decreases in each market period, see Figure 1. Such a market is called *stable*.

(Mathematically, we speak of *asymptotic stability* whenever $p_n \to p^*$ as $n \to \infty$; see Definition 1 below.)

(ii) When $m_{\mathrm{s}}/m_{\mathrm{d}} = 1$, the slope of the graph of f has magnitude one. The distance from the equilibrium point neither decreases nor increases; p_n oscillates between two values, see Figure 2. This market is still called stable.

(Mathematically, we say that the sequence p_n is a 2-cycle. The equilibrium price p^* is called stable, but not asymptotically stable.)

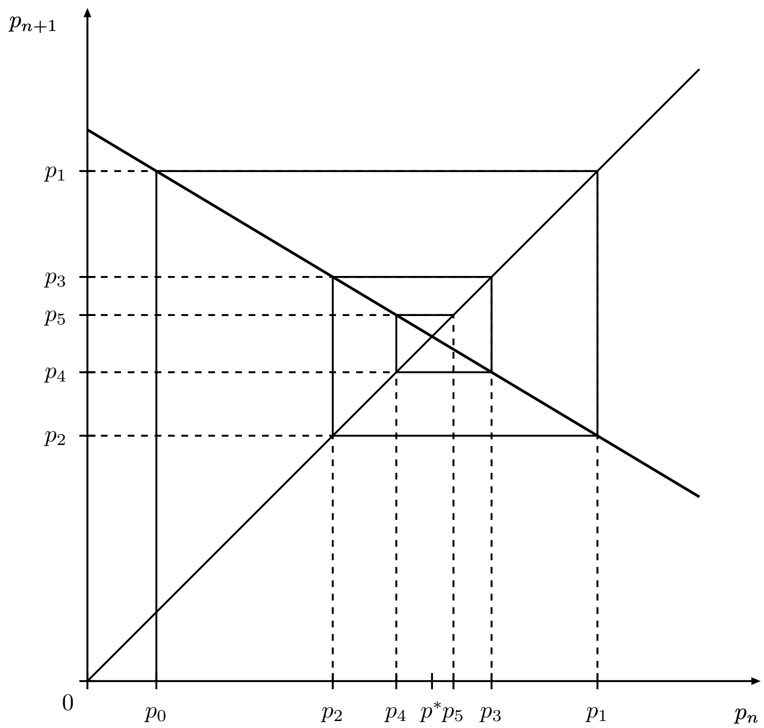

Figure 1: Asymptotically stable equilibrium price.

(iii) When $m_s/m_d = 1$, the slope of the graph of f has magnitude greater than one. The distance from the equilibrium price p^* increases in each market period, see Figure 3. Such a market is called *unstable*.

This result can be summarized as follows.

Theorem 2 (Cobweb Theorem of Economics)**.** *A market is stable if and only if the suppliers are no more sensitive to price than the consumers.*

Remark 2. We could have arrived at this conclusion more directly by noting that (19) is a linear, constant coefficient difference equation. Therefore, its solution is covered by Example 1 with $a = -m_s/m_d$ and $b = (b_d - b_s)/m_d$. The explicit solution (12) shows that p_n remains bounded as $n \to \infty$ if and only if $|a| \leq 1$, i.e. if $m_s \leq m_d$. However, the pedagogical value of going the graphical route is that it suggests how stability may be defined and verified in cases where no closed form solution is available. This is the subject of the remainder of this section.

We now proceed to discussing stability for general first order difference equations of the form

$$x_{n+1} = f(x_n)\,. \tag{22}$$

Definition 1. A number x^* is called an *equilibrium point* or *fixed point* of the difference equation (22) if

$$x^* = f(x^*)\,. \tag{23}$$

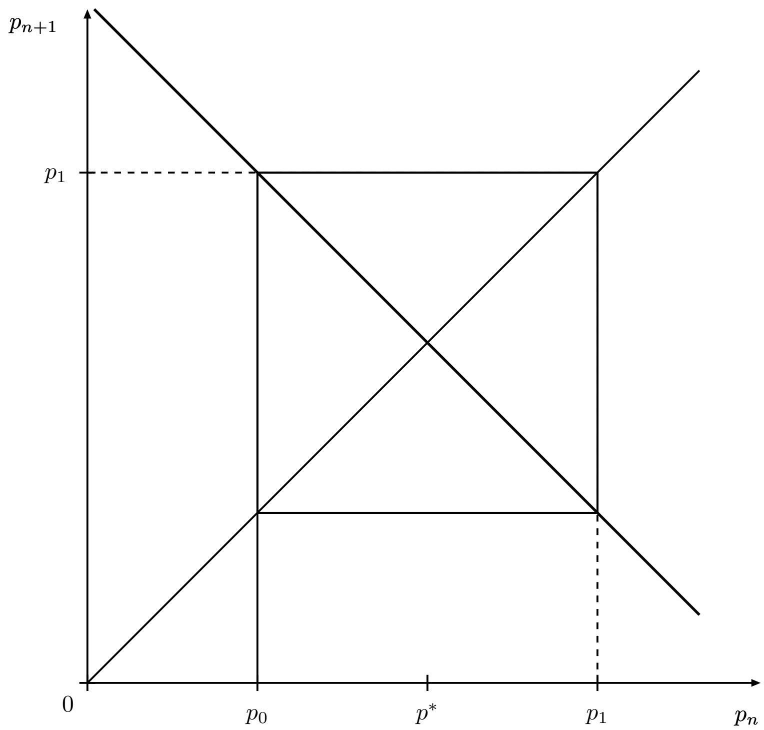

Figure 2: Stable, but not asymptotically stable equilibrium price.

Definition 2. An equilibrium point x^* of the difference equation (22) is called

(i) *stable* if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $|x_n - x^*| \leq \varepsilon$ for every $n \geq 1$ whenever $|x_0 - x^*| \leq \delta$.

(ii) *attracting* if there exists some $\eta > 0$ such that

$$\lim_{n\to\infty} x_n = x^* \tag{24}$$

whenever $|x^0 - x^*| \leq \eta$.

(iii) *asymptotically stable* if it is stable and attracting.

In other words, an equilibrium point is stable if a solution remains nearby whenever it starts sufficiently close to the equilibrium point. Asymptotic stability is a stronger property. A solution that starts sufficiently close to an asymptotically stable equilibrium point will not only remain nearby, but will actually be "attracted" by it.

Remark 3. An equilibrium point can be attracting without being stable. An example is the difference equation

$$x_{n+1} = \begin{cases} 2\,x_n & \text{if } |x_n| < 1\,, \\ 0 & \text{if } |x_n| \geq 1\,. \end{cases} \tag{25}$$

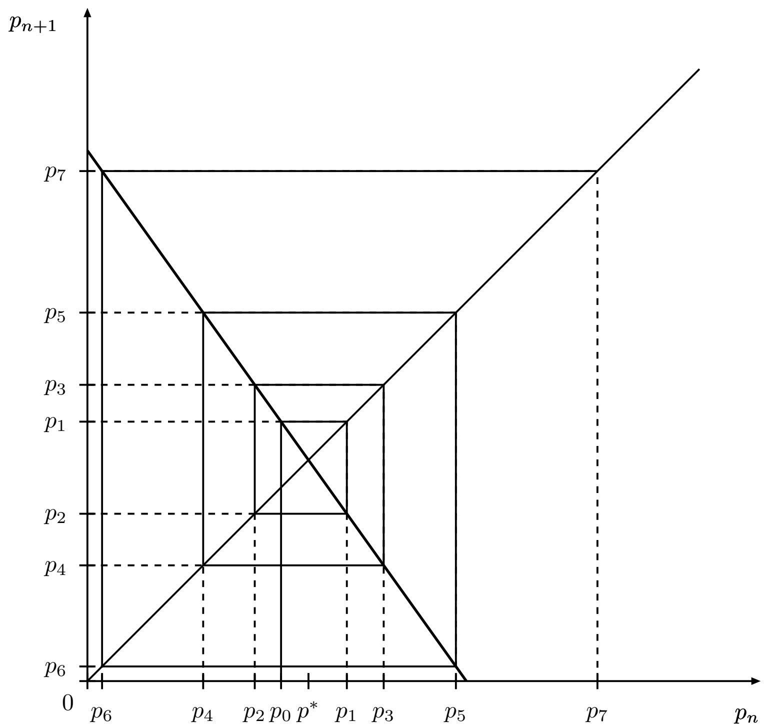

Figure 3: Unstable equilibrium price.

Here $x^* = 0$ is clearly an attracting equilibrium point, but the condition for stability fails. (Working out the details of this argument is a good exercise for those of you interested in Analysis; for our present purposes this is more a mathematical curiosity.)

In many situations, asymptotic stability is easy to verify. A first, simple result is the following.

Theorem 3. *Let x^* be an equilibrium point of the difference equation* (22) *with continuously differentiable right hand side f.*

(i) If $|f'(x^*)| < 1$, then x^* is an asymptotically stable equilibrium point.

(ii) If $|f'(x^*)| > 1$, then x^* is an unstable equilibrium point.

Proof. First, consider the case when $|f'(x^*)| < 1$. Since f' is continuous, there exist $\eta > 0$ and $L < 1$ such that $|f'(x)| < L$ for all $x \in [x^* - \eta, x^* + \eta]$. Let us assume that $x_0, \ldots, x_n$ lie in this η-neighborhood of x^*. Then

$$\begin{aligned} |x^* - x_{n+1}| &= |f(x^*) - f(x_n)| \\ &= |f'(\xi_n)| \, |x^* - x_n| \\ &\le L \, |x^* - x_n| \, , \end{aligned} \tag{26}$$

where the second step is an application of the mean value theorem which asserts the existence of some $\xi_n \in [x^*, x_n]$ such that equality holds. Since $L < 1$, estimate (26) shows that x_{n+1} is also contained in the same η-neighborhood of x^* and we can iterate the argument to conclude that

$$|x^* - x_n| \leq L^n \, |x^* - x_0| \to 0 \tag{27}$$

as $n \to \infty$. This proves part (i).

If, on the other hand, $|f'(x^*)| > 1$, by continuity of f' there exist $\eta > 0$ and $L > 1$ such that $|f'(x^*)| > L$ for all $x \in [x^* - \eta, x^* + \eta]$. Take any x_0 in this η-neighborhood of x^*. By re-tracing the argument above, we find that

$$|x^* - x_n| \geq L^n \, |x^* - x_0| \tag{28}$$

so long as x_n remains in the η-neighborhood of x^*. However, since $L > 1$, the right hand side grows without bounds as $n \to \infty$. This implies that there exists an $n \in \mathbb{N}$ such that $x_n \notin [x^* - \eta, x^* + \eta]$. Thus, x^* is not stable and we have also proved part (ii). $\square$

Remark 4. When $|f'(x^*)| = 1$, this simple test is inconclusive and a more sophisticated analysis is required. Some theorems based on higher derivatives of f are given in [1]. However, there is no easy criterion that is sufficient and necessary for stability; if standard results fail, equations have to be investigated on a case-by-case basis.

Example 3. Re-considering the market model from Problem 2 in the framework of Theorem 3, we have, from (19), that

$$f(x) = -\frac{m_\mathrm{s}}{m_\mathrm{d}} \, x + \frac{b_\mathrm{d} - b_\mathrm{s}}{m_\mathrm{d}} \tag{29}$$

and therefore

$$f'(x) = -\frac{m_\mathrm{s}}{m_\mathrm{d}} \, . \tag{30}$$

Thus, by Theorem 3, the equilibrium point p^* is asymptotically stable if $m_\mathrm{s} < m_\mathrm{d}$ and unstable if $m_\mathrm{s} > m_\mathrm{d}$. This coincides with our previous analysis.

Example 4. Consider the difference equation

$$x_{n+1} = \cos x_n \, . \tag{31}$$

Since $f(x) = \cos x$ is decreasing from 1 to 0 on the interval $[0, \pi/2]$ and $g(x) = x$ is increasing from 0 to $\pi/2$ on the same interval, the graphs of the two functions must intersect at some point $x^* \in (0, \pi/2)$. Similar arguments show that this is the only point of intersection. Thus, the difference equation has x^* as its only equilibrium point. Since $f'(x) = \sin x$, we must have $|f'(x^*)| < 1$ and we conclude that the equilibrium point is asymptotically stable. (You can try this out on your calculator by repeatedly hitting the cos-key on any input.)

4 Periodic Points and Cycles

In Problem 2 we have already encountered the situation where a solution to a difference equation alternates between two values. This concept easily generalizes to the situation when the solution returns to the starting point after cycling through a finite number of points. As in the previous section, we consider first order difference equations of the form

$$x_{n+1} = f(x_n)\,. \tag{32}$$

Definition 3. We say that a is a k-*periodic point* if it is an equilibrium point of the difference equation

$$y_{n+1} = f^k(y_n)\,. \tag{33}$$

Note that we write f^k to denote the kth composition of f with itself. Thus, $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, etc.; in this context, it is *not* the kth power of f.

Definition 4. A k-*cycle* is the orbit $\{a, f(a), \dots, f^{k-1}(a)\}$ of a k-periodic point a.

Definition 5. We say that a k-periodic point a is *stable*, *asymptotically stable*, or *unstable* if a is a stable, asymptotically stable, or unstable equilibrium point of (33), respectively.

Remark 5. It can be shown that all the k-periodic points in a k-cycle have the same stability property. Thus, we can say "k-cycle" instead of "k-periodic point" in Definition 5.

Remark 6. It may happen that an orbit starting at some initial point b becomes periodic only after a finite number of steps. Such orbits are called *eventually periodic.*

Since the discussion of periodic points can be reduced to the discussion of equilibrium points for the map (33), everything we have said about stability of equilibrium points naturally translates into results on the stability of cycles. For future reference, we state the following elementary stability criterion for 2-cycles.

Theorem 4. *Suppose that the difference equation* (32) *has a* 2*-cycle* $\{a, b\}$ *and that* f *is continuously differentiable. Then*

(i) the 2*-cycle is asymptotically stable if* $|f'(a)\,f'(b)| < 1$,

(ii) the 2*-cycle is unstable if* $|f'(a)\,f'(b)| > 1$.

The proof is a direct application of Theorem 3 to (33). The details are left as an exercise to the reader.

5 The logistic equation

Difference equations are frequently used to model phenomena of population growth, spread of disease, or market dynamics as in the example of the previous section. One of the simplest such population models is the *logistic difference equation*, which we introduce in the following.

Problem 3. Describe the growth of a population of a single species in a habitat of constrained resources.

Let us discuss a set of modeling assumptions that allow us to obtain a fully specified, yet simple quantitative description. As for the market model of Section 3, we assume discrete periods of equal length with regard to physical time, labeled with integer n. The size of the population in period n is denoted p_n.

When resources are plenty, the number of offspring per model period will be a fraction b of the size of the population in existence. Likewise, the number of deaths per model period will be another fraction d of the population size. Thus,

$$p_{n+1} = (1 + b - d)\, p_n \equiv \mu\, p_n \,. \tag{34}$$

Since this equation describes good, unconstrained conditions, we expect that $b > d$ and therefore $\mu > 1$. The solution to (34),

$$p_n = \mu^n\, p_0 \,, \tag{35}$$

then grows without bounds as $n \to \infty$.

A model for the growth of a population in a habitat of constrained resources must therefore contain a term that limits the growth when the population size becomes large. There are many effects that may be taken into account and models can become very complicated. However, with the following simplifying assumptions, we may obtain a very simple, yet powerful model. To facilitate the discussion, we assume that the habitat consists of s discrete sites. (This assumption is not essential; we would reach the same conclusion in a corresponding continuum set-up.) We further assume that

(i) In each model period, there is a loss of population, for example due to fights or starvation, proportional to the probability that two or more individuals occupy the same site.

(ii) Individuals move through the habitat at random, i.e. each individual is equally likely to occupy each site.

(iii) The habitat is large and sparsely populated. More precisely, we assume that $1 \ll p \ll \sqrt{s}$.

Thus, it is our task to compute the probability of two individuals occupying the same site, and to simplify the resulting expression according to assumption (iii).

We recognize that the problem is precisely the "birthday problem" we discussed earlier. As in the solution of the birthday problem, we first compute the probability that no two individuals occupy the same site,

$$
\begin{aligned}
P(\text{no two on same site}) &= \frac{s \cdot (s-1) \cdot \cdots \cdot (s-p+1)}{s^p} \\
&= \frac{s!}{s^p\,(s-p)!} \\
&\sim \frac{\sqrt{2\pi s}\, s^s\, \mathrm{e}^{-s}}{s^p\,\sqrt{2\pi(s-p)}\,(s-p)^{s-p}\,\mathrm{e}^{-(s-p)}} \\
&= \sqrt{\frac{s}{s-p}}\, \exp\biggl((p-s)\, \ln\frac{s-p}{s} - p\biggr) \qquad (36)
\end{aligned}
$$

We now approximate this expression up to first order terms in the small quantity p/s. We use two-term Taylor approximations of the functions

$$
\ln(1+x) \approx x - \frac{1}{2}\,x^2\,, \quad \mathrm{e}^x \approx 1+x\,, \quad \frac{1}{1-x} \approx 1+x\,, \quad \text{and} \quad \sqrt{1+x} \approx 1 + \frac{1}{2}\,x\,. \qquad (37)
$$

Then

$$
\ln\frac{s-p}{s} = \ln\biggl(1 - \frac{p}{s}\biggr) \approx -\frac{p}{s} - \frac{1}{2}\,\frac{p^2}{s^2}\,, \qquad (38)
$$

so that, since $1 \gg p^2/s \gg p^3/s^2$,

$$
\exp\biggl((p-s)\,\ln\frac{s-p}{s} - p\biggr) \approx \exp\biggl((s-p)\,\biggl(\frac{p}{s} + \frac{1}{2}\,\frac{p^2}{s^2}\biggr) - p\biggr) \approx \exp\biggl(-\frac{p^2}{2s}\biggr) \approx 1 - \frac{p^2}{2s}\,. \qquad (39)
$$

Further,

$$
\sqrt{\frac{s}{s-p}} = \sqrt{\frac{1}{1-p/s}} \approx \sqrt{1+\frac{p}{s}} \approx 1 + \frac{1}{2}\,\frac{p}{s}\,. \qquad (40)
$$

Altogether, noting that $1 \gg p^2/s \gg p/s \gg p^3/s^2$,

$$
P(\text{no two on same site}) \approx \biggl(1 + \frac{1}{2}\,\frac{p}{s}\biggr) \cdot \biggl(1 - \frac{p^2}{2s}\biggr) \approx 1 - \frac{p^2}{2s} \qquad (41)
$$

and therefore

$$
P(\text{at least two on same site}) = 1 - P(\text{no two on same site}) \approx \frac{p^2}{2s}\,. \qquad (42)
$$

We conclude the following.

Theorem 5. *The probability that at least two of p randomly placed individuals share the same site in a large sparsely populated habitat is asymptotically proportional to p^2.*

Theorem 5 suggests that we should augment the unconstrained growth equation (34) by a loss term proportional to p_n^2. The resulting difference equation is the *logistic difference equation*

$$p_{n+1} = \mu\, p_n - \nu\, p_n^2 \tag{43}$$

with positive parameters μ and ν. The qualitative behavior is essentially governed by a single parameter. Indeed, setting $p_n = \mu/\nu\, x_n$, we obtain the logistic equation in its standard form

$$x_{n+1} = \mu\, x_n\, (1 - x_n)\,. \tag{44}$$

Remark 7. In this model we allow the number of individuals to be any positive real number. When the number of individuals is large, this is a useful simplification. However, when the number of individuals is small, the discreteness of p becomes significant. Moreover, in real population dynamics, random fluctuations can have dramatic impact on small populations.

Remark 8. If we reconsider the derivation leading to Theorem 5 carefully, we see that the constant ν must be small and inversely proportional to the artificial parameter s. For purposes of this discussion, let $\nu = 1/s$. Then a nontrivial equilibrium point of the logistic difference equation (43) is $p^* = s\,(\mu - 1)$. (A more detailed discussion of equilibrium points is given in the next section.) Since the equilibrium point should not depend on the artificial parameter s, we must therefore require that μ be very close to 1 as s becomes large. This corresponds to choosing units of time where one iteration of the logistic map corresponds to a very small amount of real time.

Remark 9. The statement of Remark 8 can also be re-interpreted as follows. Choose the arbitrary normalization $p^* = 1$, so that $\mu = 1 + 1/s$. Then (43) coincides with the so-called explicit Euler scheme [5] for the numerical solution of the *logistic differential equation*

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p - p^2\,, \tag{45}$$

commonly written in the form

$$p_{n+1} = p_n + h\,(p_n - p_n^2)\,, \tag{46}$$

where $h = 1/s$ is the so-called step size. In other words, our derivation of the model is valid precisely in the parameter regime where the logistic difference equation is a good approximation to the logistic differential equation.

6 Bifurcations

In the following, we study the qualitative behavior of the first order difference equations of the form (23) as a function of a parameter. By *qualitative behavior* we mean the number of equilibrium points, of k-cycles, and their stability. We speak of a *bifurcation* when a small change in the parameter causes a change in the qualitative behavior of the equation.

We illustrate the concept using the logistic equation in the form (44) where μ is the parameter we will vary.

Equilibrium points. We first determine the equilibrium points of the logistic difference equation. Setting

$$f(x) = \mu\, x\,(1 - x)\,, \tag{47}$$

the fixed point equation $x = f(x)$ has two solutions

$$x_1^* = 0 \quad \text{and} \quad x_2^* = \frac{\mu - 1}{\mu}\,. \tag{48}$$

Since $f'(x) = \mu - 2\,\mu\, x$, we have, in particular,

$$f'(x_1^*) = \mu \quad \text{and} \quad f'(x_2^*) = 2 - \mu\,. \tag{49}$$

By Theorem 3,

- x_1^* is an asymptotically stable equilibrium point for $|\mu| < 1$ and an unstable equilibrium point for $|\mu| > 1$.
- x_2^* is an asymptotically stable equilibrium point for $1 < \mu < 3$ and an unstable equilibrium point for $\mu < 1$ or $\mu > 3$.

The points $\mu = 1$ and $\mu = 3$ are bifurcation points since the stability of the equilibrium points is changing as the parameter μ varies across.

2-cycles. It can be shown (homework exercise!) that the logistic map has a 2-cycle whenever $\mu > 3$ with orbit $\{a, b\}$ where

$$a = \frac{1 + \mu - \sqrt{(\mu - 3)(\mu + 1)}}{2\,\mu}\,, \tag{50}$$

$$b = \frac{1 + \mu + \sqrt{(\mu - 3)(\mu + 1)}}{2\,\mu}\,. \tag{51}$$

$$\tag{52}$$

Moreover, by Theorem 4, this 2-cycle is asymptotically stable for $3 < \mu < 1 + \sqrt{6}$ and is unstable for $\mu > 1 + \sqrt{6}$.

Thus, not only does the equilibrium point x_2^* lose stability at $\mu = 3$, but a stable 2-cycle emerges. There is a third bifurcation point at $\mu = 1 + \sqrt{6}$ where the 2-cycle loses stability and, as can be shown by lengthy calculation, a stable 4-cycle emerges.

The behavior analyzed so far is summarized in Figure 4. This graph is called a *bifurcation diagram*. The horizontal axis shows the parameter while the vertical axis shows the location of equilibrium points and cycles. In standard terminology, the bifurcation at $\mu = 1$ is called a *transcritical bifurcation* and the bifurcation at $\mu = 3$ a *pitchfork bifurcation*.

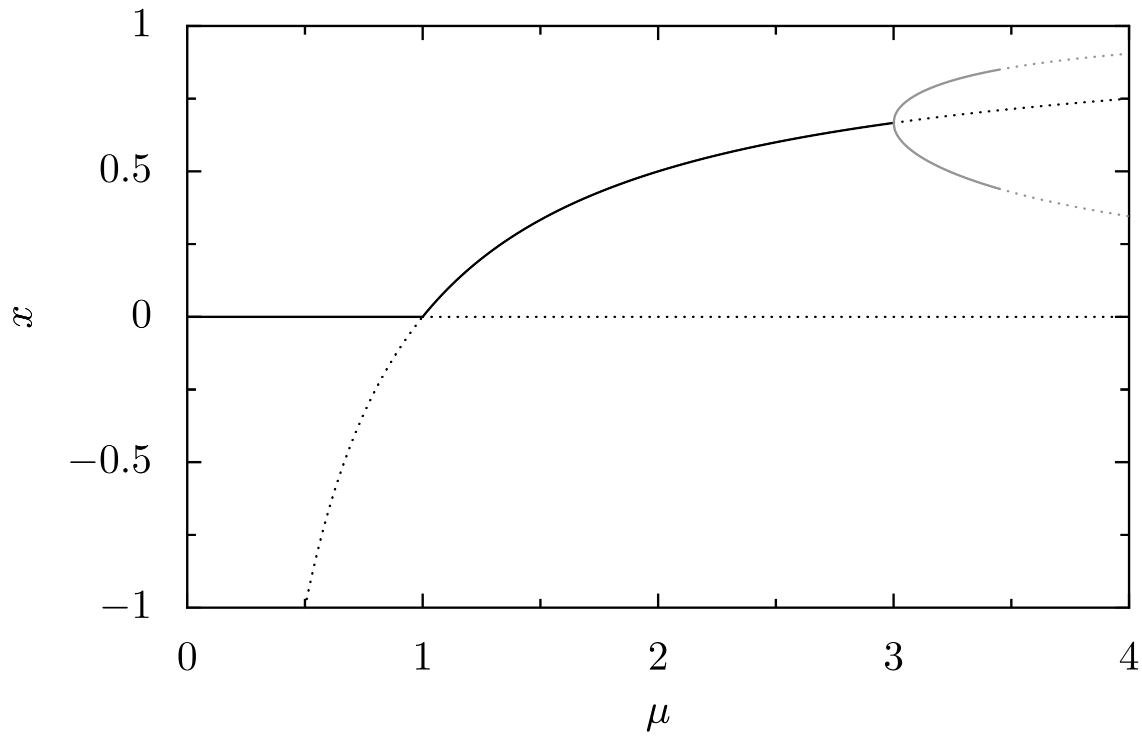

Figure 4: Partial bifurcation diagram for the logistic map. Shown are the branches which we have explicitly computed. Unstable branches are dotted, stable branches indicated by solid lines. The equilibrium points are in black, the 2-cycle in gray. At the point where the 2-cycle becomes unstable, a stable 4-cycle emerges. This 4-cycle and its further bifurcations are *not* shown.

Higher cycles and chaos. The analysis of the bifurcation structure beyond the 2-cycle is rather complicated; [4] has some more information and references to the research literature. In particular, there are no closed form expressions for most branches of the bifurcation diagram.

On the other hand, it is comparatively easy to find the stable cycles by numerical computation. For each value of μ, choose an arbitrary starting point and iterate the logistic map a large number of times. Since an asymptotically stable cycle is attracting, the value of x_n for n large will now be very close to the stable cycle. Keep iterating and plot the next many iterates—they will now approximately trace out the stable cycle in the bifurcation diagram. Figure 5 shows the result of such a computation (with the additional twist that the graph is shaded according to the probability of a certain pixel being visited during the iteration).

Note that for each initial value you will approach exactly one stable equilibrium point or cycle. For the logistic map it is known that only one starting point will suffice. In general, additional analysis or systematic search is required to verify that all possible attracting sets have been found.

Through a combination of analysis and numerical computation, the following picture has

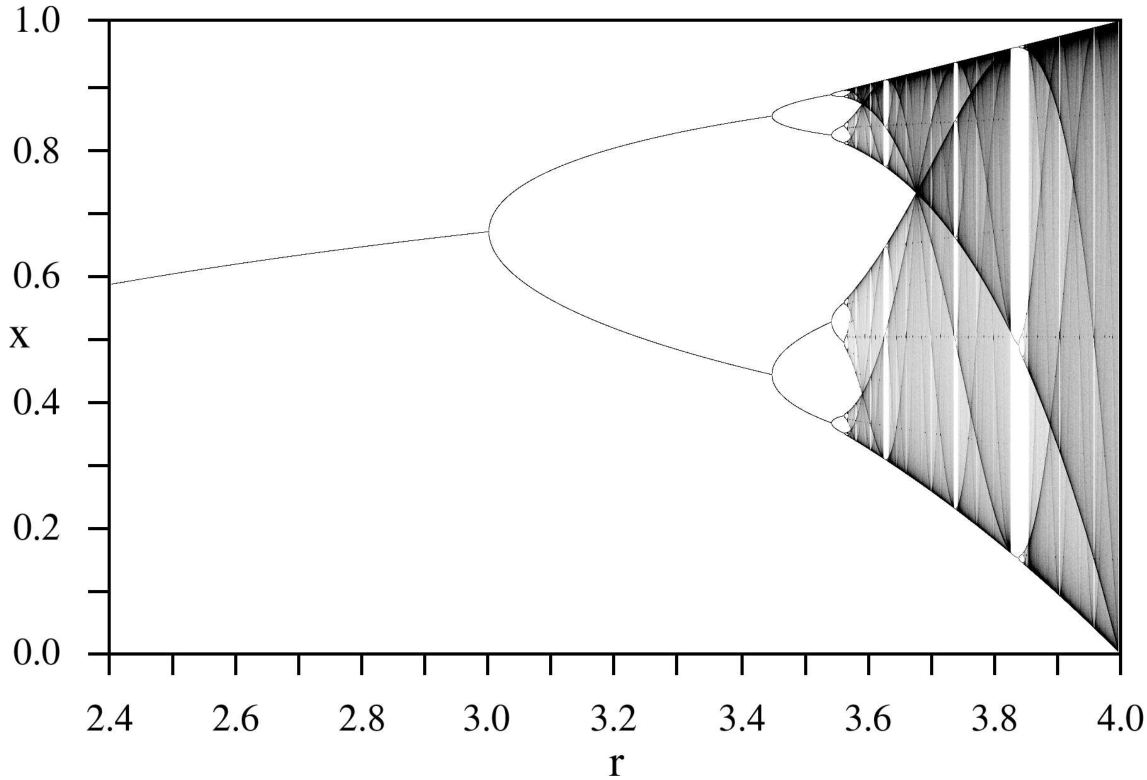

Figure 5: Numerically computed bifurcation diagram for the logistic map. The computation can only show the attracting sets. The graph is taken from [6] where the method of computation is explained in detail. Their parameter r is the same as μ in (44).

been established [3, 4, 6].

- As μ is increased, each 2^k-cycle has a bifurcation point where it becomes unstable and bifurcates into a stable 2^{k+1}-cycle.

  The first such bifurcation point is the break-up of the stable equilibrium point or, equivalently, 2^0-cycle into the 2^1-cycle at $\mu = 3$ as shown in Figure 4. All other such bifurcation points are qualitatively similar.

- If ℓ_k denote the interval on the μ-axis in which the 2^k cycle is stable, then

$$\lim_{k\to\infty} \frac{\ell_k}{\ell_{k+1}} = \delta = 4.6692016\ldots . \tag{53}$$

  This number is called the *Feigenbaum constant* and is universal across large classes of maps. In particular, the fact that this limit exists implies that this cascade of bifurcations, often referred to as a *period doubling cascade*, takes place in a finite interval on the μ-axis. For the logistic map, the cascade ends at $\mu \approx 3.57$, as can be seen in Figure 5.

- For $3.57 \lesssim \mu < 4$, cycles of all periods exist. For most values of μ, all cycles are unstable and we speak of *chaos*. However, in the interior of this interval there are *islands of stability* in which stable $3, 5, \ldots$-periodic cycles exist, each with their own period doubling cascade.

- Beyond $\mu = 4$, solutions diverge for almost all initial values.

We close this section with a short remark on chaos. Mathematical definitions of "chaos" are somewhat problem dependent, but usually involve the following features.

- *Sensitive dependence on initial conditions.* More precisely, if the system is started from any two nearby initial points, then the distance between the resulting solutions grows exponentially in time, or with the number of iterations, until it is on the order of the system size.

- *Dense orbits.* Every orbit of the system comes arbitrarily close to every point in the domain of definition, or at least in a "massive" subset of the domain of definition, called a *strange attractor*.

The notion of chaos is important for a number of reasons. First, it is a mathematically challenging task to describe and quantify the behavior of chaotic systems in meaningful ways. A lot of new and important mathematics has emerged from the pursuit of such questions. Second, it shows that very simple systems can have very complicated behavior—the logistic difference equation being a case in point. Third, the presence of chaotic dynamics implies severe practical limits to the predictive power of a model. Since orbits diverge and mix at an exponential rate, doubling the accuracy of the initial values (which are typically measurements) will not double the time interval over which an accurate predication can be achieved, but only increase this time interval by a fixed amount. This severely restricts the ability to achieve accurate long-range predictions.

7 Generating functions revisited

Let us now consider difference equations of order greater than one. Earlier this semester, we discussed the method of generating functions for the solution of certain difference equations. In the following, we review the method and state the theory from a different viewpoint.

In this section, we assume that the difference equation is *linear* and *homogeneous* with *constant coefficients*. For simplicity, we will only consider *second order* equations, although everything we do will generalizes to higher order in a natural way. Such an equation can be written in the form

$$a_{n+1} + c_1\, a_n + c_2\, a_{n-1} = 0\,; \tag{54}$$

for the time being, we leave the initial conditions unspecified. Now define the generating function

$$\phi(x) = \sum_{i=0}^{\infty} a_i\, x^i\,. \tag{55}$$

Then, by expanding the left hand expression and ordering terms by powers in x, we find that

$$(1 + c_1\,x + c_2\,x^2)\,\phi(x) = a_0 + (a_1 + c_1\,a_0)\,x\,, \tag{56}$$

or

$$\phi(x) = \frac{a_0 + (a_1 + c_1\,a_0)\,x}{1 + c_1\,x + c_2\,x^2}\,. \tag{57}$$

Thus, we need to find the explicit power series expansion of (57) whose coefficients are the solution to the difference equation (54). Let us recall that, in the ring of formal power series (cf. [2], p. 22),

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \tag{58}$$

and

$$\frac{1}{(1-x)^2} = 1 + 2\,x + 3\,x^2 + 4\,x^3 + \dots\,. \tag{59}$$

Let us denote the roots of the polynomial $1 + c_1\,x + c_2\,x^2$ by μ_1 and μ_2. (In the field of complex numbers, these roots always exist. In the following, we will consider examples with real roots only; the general discussion, however, applies equally to the complex case.) We must distinguish two cases.

Two distinct roots. This is the case we encountered previously, e.g. when determining the general form of the Fibonacci sequence. In this case, we know that ϕ has a partial fraction decomposition of the form

$$\begin{aligned}
\phi(x) &= \frac{A'}{\mu_1 - x} + \frac{B'}{\mu_2 - x} \\
&= \frac{A}{1 - \dfrac{x}{\mu_1}} + \frac{B}{1 - \dfrac{x}{\mu_2}} \\
&= A\left(1 + \frac{x}{\mu_1} + \frac{x^2}{\mu_1^2} + \dots\right) + B\left(1 + \frac{x}{\mu_2} + \frac{x^2}{\mu_2^2} + \dots\right),
\end{aligned} \tag{60}$$

where A and B are some constants which depend on a_0 and a_1 and which are determined by the partial fraction decomposition, and where $A' = A\,\mu_1$ and $B' = B\,\mu_2$. We now take the point of view that, although we could work out the values of the constants through the computation above, we simply note that this computation shows that the general form of the solution to the difference equation (54) is

$$a_n = A\,\mu_1^{-n} + B\,\mu_2^{-n}\,. \tag{61}$$

The constants A and B can then be determined by the condition that the initial conditions are replicated by (61).

Double Roots. When the two roots $\mu_1 = \mu_2 \equiv \mu$ coincide—we speak of a *double root*—a partial fraction decomposition of the form (60) is not possible. Instead, we refer to (59) and write

$$\begin{aligned}\phi(x) &= \frac{A'\,(\mu - x) + B'}{(\mu - x)^2} \\ &= \frac{A}{1 - \dfrac{x}{\mu}} + \frac{B}{\left(1 - \dfrac{x}{\mu}\right)^2} \\ &= A\left(1 + \frac{x}{\mu} + \frac{x^2}{\mu^2} + \ldots\right) + B\left(1 + 2\,\frac{x}{\mu} + 3\,\frac{x^2}{\mu^2} + \ldots\right). \end{aligned} \tag{62}$$

As in the case of two distinct roots, A and B are two constants which depend on a_0 and a_1. In principle, they could be determined by explicitly carrying all constants through the computation above. Instead, however, we only note that the general form of the solution to the difference equation (54) is

$$a_n = A\,\mu^{-n} + B\,(n+1)\,\mu^{-n}\,. \tag{63}$$

As before, the constants A and B can be determined by the condition that the initial conditions are replicated by (63).

We have now effectively proved a theorem on the general solution of (54). We can rewrite the final result in a form more commonly encountered by making the following modifications to the expressions above. First, notice that we can make the prefactor of the second term in (63) proportional to n rather than $n+1$. The "missing" bit can then be grouped together with the first term, forcing a redefinition of the constant A. Second, μ is a root of $1 + c_1\,x + c_2\,x^2$ if and only if $\lambda = 1/\mu$ is a root of $x^2 + c_1\,x + c_2$. We can therefore express the solution in terms of roots of the latter polynomial. Finally, we can rescale the difference equation (54) by an arbitrary constant without changing any of the above procedure. With these modifications, we restate our result as follows.

Definition 6. The *characteristic polynomial* of the difference equation

$$c_0\,a_{n+1} + c_1\,a_n + c_2\,a_{n-1} = 0 \tag{64}$$

is the polynomial

$$p(x) = c_0\,x^2 + c_1\,x + c_2\,. \tag{65}$$

Theorem 6. *Let λ_1 and λ_2 denote the roots of the characteristic polynomial* (65)*. Then any solution to* (64) *is of the form*

(i) when $\lambda_1 \neq \lambda_2$, then $a_n = A\,\lambda_1^n + B\,\lambda_2^n$,

(ii) when $\lambda_1 = \lambda_2 \equiv \lambda$, then $a_n = A\,\lambda^n + B\,n\,\lambda^n$,

for arbitrary constants A and B.

Remark 10. Strictly speaking, our derivation breaks down when one of the roots is zero. For a true second order equation, however, $c_2 \neq 0$ and this cannot happen. Otherwise we are back to the case of first order equations discussed in Section 2.

Remark 11. *Linear Algebra* provides an alternative way to prove Theorem 6. The difference equation can be written as a system of two first-order difference equations. It turns out that the characteristic polynomial of the difference equation equals the characteristic polynomial of the resulting matrix. When the roots are distinct, the system can be solved by diagonalization; in the case of a double root, the Jordan normal form is used to solve the equation.

We now give two typical application examples of the theory. The first example also illustrates that Theorem 6 is particularly useful when the difference equation is augmented not by initial conditions, but by some other conditions on the solution. In this case, a straightforward application of the method of generating functions would not be possible.

Problem 4 (Gambler's ruin)**.** A gambler is repeatedly playing a game in which he is to gain 1 Euro with probability q and lose 1 Euro with probability $1-q$. Assume that the gambler has n Euros to play. When he is down to zero, we say that the gambler is ruined and cannot continue gambling. He will also quit when he has reached a pre-set target of N Euros. What is the probability that the gambler is ruined?

Let p_n denote the probability of being ruined when owning n Euros. If $n = 0$, the gambler is ruined for sure, thus $p_0 = 1$. When $n = N$, the gambler quits without being ruined, so $p_N = 0$. For $0 < n < N$, the gambler keeps playing. If he wins, he will be ruined with probability p_{n+1} in the next iteration of the game; if he loses, he will be ruined with probability p_{n-1}. Since the first alternative occurs with probability q and the second alternative occurs with probability $1-q$, we deduce that

$$p_n = q\, p_{n+1} + (1-q)\, p_{n-1} \,. \tag{66}$$

This is clearly a difference equation of the form (64) with characteristic polynomial

$$p(x) = q\, x^2 - x + (1-q) \tag{67}$$

whose roots are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4\, q\,(1-q)}}{2\, q} = \frac{1 \pm \sqrt{(1-2\, q)^2}}{2\, q} \tag{68}$$

so that

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = \frac{1-q}{q} \,. \tag{69}$$

We can now apply Theorem 6. There are two distinct cases.

Fair game. When the probability of losing and of winning the game is both $\frac{1}{2}$, we speak of a *fair game*. In this case, both roots of the characteristic polynomial equal $\lambda = 1$, so that the general solution to (66) is

$$p_n = A + B\,n\,. \tag{70}$$

Since $p_0 = 1$, we must have $A = 1$. Then $p_N = 0$ gives $1 + B\,N = 0$ so that $B = -1/N$ and therefore

$$p_n = 1 - \frac{n}{N}\,. \tag{71}$$

In particular, if the gambler owns a finite amount of money n, but does not quit gambling, we can represent this by taking the limit $N \to \infty$. In this case, the gambler will be ruined with probability 1.

Biased game. When $q \neq \frac{1}{2}$, the two roots in (69) are distinct. In this case,

$$p_n = A + B \left(\frac{1-q}{q}\right)^n . \tag{72}$$

Since $p_0 = 1$, we must have $A + B = 1$. Moreover, since $p_N = 0$, we find that

$$A + B \left(\frac{1-q}{q}\right)^N = 0\,. \tag{73}$$

Solving for A and B, we obtain

$$A = -\frac{\left(\frac{1-q}{q}\right)^N}{1 - \left(\frac{1-q}{q}\right)^N} \qquad \text{and} \qquad B = \frac{1}{1 - \left(\frac{1-q}{q}\right)^N}\,. \tag{74}$$

Inserting this result back into (72), we finally obtain

$$p_n = \frac{\left(\frac{1-q}{q}\right)^n - \left(\frac{1-q}{q}\right)^N}{1 - \left(\frac{1-q}{q}\right)^N}\,. \tag{75}$$

Let us again ask the question what happens when the gambler does not quit gambling, i.e. take the limit $N \to \infty$ with n fixed. If $q < \frac{1}{2}$, then $(1-q)/q > 1$ so that

$$p_n(N) \to 1 \quad \text{as } N \to \infty\,, \tag{76}$$

i.e., the gambler is ruined for sure; if $q > \frac{1}{2}$, then $(1-q)/q < 1$ so that

$$p_n(N) \to \left(\frac{1-q}{q}\right)^n \quad \text{as } N \to \infty\,. \tag{77}$$

We see that in a game which is biased in favor of the gambler, there is non-zero probability of "survival" even if he will never quit.

We summarize these results as follows.

Theorem 7 (Gambler's theorem)**.** *If you don't quit playing a game that's not in your favor, you will be ruined for sure.*

We close this section with an example from [3] which illustrates interpretation and possible qualitative behavior of a second order linear difference equation or, equivalently, of a two-variable first order system of difference equations.

Problem 5 (Romeo and Juliet)**.** Model the onset of a love/hate relationship between Romeo and Juliet, taking into account

(i) the change of feeling for each other while apart, and

(ii) the change of feeling for each other in response to the other's emotions.

Again, we construct a discrete time model with n denoting units of time, say days. Romeo's feelings for Juliet on day n are quantified by the variable R_n, positive values representing love and negative values representing hate. Similarly, Juliet's feelings for Romeo are quantified by the variable J_n. Before Romeo and Juliet meet on day $n = 0$, their feelings for each other are neutral, so $J_0 = R_0 = 0$. We will first formulate a simple model for the dynamics of their relationship and then investigate whether zero is a stable equilibrium point, or whether a small perturbation may evolve into something better (or worse).

When Romeo and Juliet are on their own, R_n and J_n evolve independent of each other. Let us therefore focus on R_n only. We assume a simple linear model,

$$R_{n+1} = a_R\, R_n \tag{78}$$

where a_R is a positive parameter. If a_R was negative, Romeo's feeling toward Julia would alternate from love to hate on consecutive days, which is not what we would expect. When $0 < a_R < 1$, then Romeo's intrinsic love for Juliet fades out, when $a_R > 1$, it increases with time.

In the presence of Julia, Romeo also reacts to her feelings toward him. This feedback is modeled by a second term in Romeo's equation,

$$R_{n+1} = a_R\, R_n + p_R\, J_n\,, \tag{79}$$

where the feedback coefficient p_R can be either positive or negative. The different possible combinations of a_R and p_R give rise to four romantic styles:

	$p_R < 0$	$p_R > 0$
$0 < a_R < 1$	hermit	cautious lover
$a_R > 1$	likes to tease, but not to please	eager beaver

The equation for Juliet's feeling toward Romeo shall be symmetric to Romeo's equation, so that the combined system of difference equations reads

$$R_{n+1} = a_R\, R_n + p_R\, J_n\,, \tag{80a}$$
$$J_{n+1} = a_J\, J_n + p_J\, R_n\,. \tag{80b}$$

We can now investigate the outcome of their love affair by studying these difference equations using Theorem 6. The crucial observation is that a system of two difference equations can always be formulated as a single second order difference equation, and vice versa. Instead of proving this general statement, we proceed by example. Shifting the index of the first equation and multiplying the second by p_R, we obtain

$$R_{n+2} = a_R\, R_{n+1} + p_R\, J_{n+1}\,, \tag{81a}$$
$$p_R\, J_{n+1} = p_R\, a_J\, J_n + p_R\, p_J\, R_n\,, \tag{81b}$$

so that

$$R_{n+2} = a_R\, R_{n+1} + p_R\, a_J\, J_n + p_R\, p_J\, R_n \tag{82}$$

The remaining J_n can be eliminated by using the unshifted version of (80a), and we finally obtain

$$R_{n+2} - (a_R + a_J)\, R_{n+1} + (a_J\, a_R - p_J\, p_R)\, R_n = 0\,. \tag{83}$$

The characteristic polynomial of this difference equation is

$$p(x) = x^2 - (a_R + a_J)\, x + a_J\, a_R - p_J\, p_R \tag{84}$$

which has roots

$$\begin{aligned} \lambda_{1,2} &= \frac{a_J + a_R \pm \sqrt{(a_R + a_J)^2 - 4\,(a_J\, a_R - p_J\, p_R)}}{2} \\ &= \frac{a_J + a_R \pm \sqrt{(a_R - a_J)^2 + 4\, p_J\, p_R}}{2}\,. \end{aligned} \tag{85}$$

A comprehensive analysis of the solution is rather involved and best done in the linear algebra framework (as this would allow to make simultaneous statements about Romeo's and Juliet's feelings). However, we can make a few basic observations from (85):

- When a_R and a_J are similar in magnitude and p_J and p_R have opposite signs (i.e. exactly one of the two counter-reacts to the other's feelings), then the expression under the square root may be negative. In this case, the difference equation can still be solved, but the roots are complex numbers which manifests itself in oscillatory behavior of the solutions. The relationship then gets into a love-hate cycle which may grow or decay in amplitude; see the right hand graph in Figure 6.

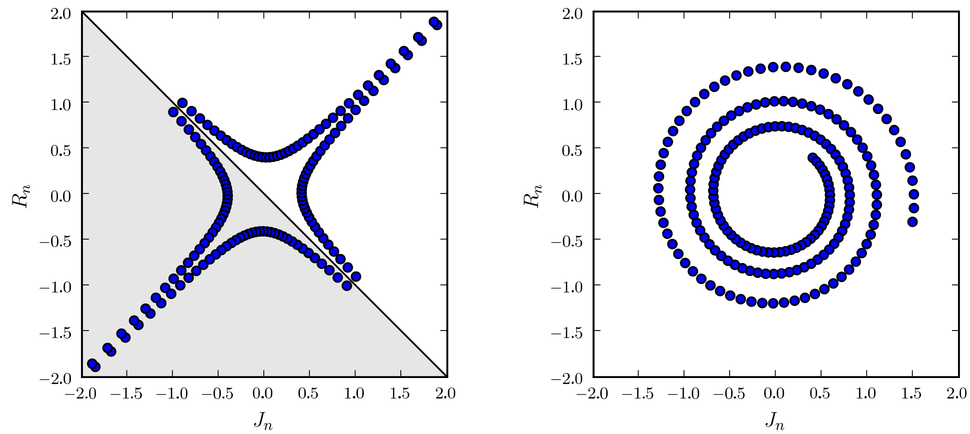

Figure 6: The Romeo-Juliet system with neutral self-induced growth coefficients $a_R = a_J = 1$ and positive coupling coefficients $p_R = p_J = 0.1$ (left) vs. mixed coupling coefficients $p_R = -0.1$, $p_J = 0.1$ (right). The shaded region in the left hand graph indicates initial configurations which evolve into growing mutual hate. On the right, the relationship evolves into a love-hate cycle independent of the initial conditions.

- A relationship can only develop if at least one of the roots is greater than one, so that the general solution from Theorem 6 has at least one growing component.
- We cannot infer from (83) that if Romeo's love for Juliet is growing that Juliet's love for Romeo is also growing. We need to explicitly look at the corresponding Juliet solution.
- Initialization of this model is very important. In a parameter regime where love can grow, hate can also grow. As the left graph in Figure 6 shows, initial configurations which are close can result in completely different outcomes.
- This model can only represent the onset of an affair, as there are no limiting terms which would keep initially growing solutions bounded as $n \to \infty$. This situation is very much like for the simple, unconstrained model (34) as opposed to the full logistic equation for the dynamics of a one-species population.

8 Inhomogeneous linear equations

The theory developed in the previous section may seem rather special. However, there are two important principles that show that the study of linear homogeneous problems is relevant in a much more general context.

The first principle concerns the general solution to inhomogeneous linear equations of the form (2), and will be formulated below. The next principle concerns the study of equilibrium points to nonlinear equations and will be discussed in Section 9 below.

To formulate the statement in succinct terms, we need to introduce the concept of linear independence. Let $x_n^{(1)}, \ldots, x_n^{(j)}$ denote j sequences (for example, j solutions to a difference equation with different starting values). Then we say that $x_n^{(1)}, \ldots, x_n^{(j)}$ are *linearly independent* or simply *independent* provided that

$$\alpha_1 \, x_n^{(1)} + \cdots + \alpha_j \, x_n^{(j)} = 0 \tag{86}$$

for every $n \in \mathbb{N}$ implies that $\alpha_1 = \cdots = \alpha_j = 0$. Without discussing the geometric content of this statement (this can be found in any book on Linear Algebra), we simply note that in the case of $j = 2$ sequences, the condition for independence reduces to the simple statement that one sequence is not a multiple of the other.

Recall the inhomogeneous linear equation (2),

$$x_{n+1} = a_0(n) \, x_n + a_1(n) \, x_{n-1} + \cdots + a_{k-1}(n) \, x_{n-k+1} + b(n) \, . \tag{87}$$

We say that the equation

$$x_{n+1} = a_0(n) \, x_n + a_1(n) \, x_{n-1} + \cdots + a_{k-1}(n) \, x_{n-k+1} \, . \tag{88}$$

is the *associated homogeneous* equation. Then the following is true.

Theorem 8 (Superposition Principle)**.** *Let p_n solve the inhomogeneous equation* (87) *for some arbitrary starting values. Further, let $y_n^{(1)}, \ldots, y_n^{(k)}$ solve the associated homogeneous equation* (88)*. Set*

$$x_n = p_n + \alpha_1 \, y_n^{(1)} + \cdots + \alpha_n \, y_n^{(k)} \tag{89}$$

for some $\alpha_1, \ldots, \alpha_k$. Then

(i) x_n is a solution to the inhomogeneous equation (87)*.*

(ii) Vice versa, any solution to (87) *can be written in the form* (8) *provided that $y_n^{(1)}, \ldots, y_n^{(k)}$ are independent.*

Part (i) is easily proved by direct substitution of (89) into (87). We leave this as an exercise to the reader. The proof of part (ii) involves nontrivial Linear Algebra and shall therefore be omitted.

When the coefficients $a_1, \ldots, a_k$ depend on n, this theorem, though true and important for the development of the theory, is of not much practical help. In fact, it can be argued that such equations are really nonlinear "in disguise" and that we therefore should not expect to be able to develop a universal solution procedure.

If, on the other hand, the coefficients $a_1, \ldots, a_k$ are independent of n, then Theorem 6 (respectively its generalization to order $k > 2$) provides a method for solving the homogeneous problem in a systematic way. So one only needs to guess a single inhomogeneous solution, which is often possible, to automatically obtain *all* solution to the inhomogeneous equation.

Example 5 (Model for GDP growth)**.** An idealized model for the evolution of the gross domestic product (GDP) of a country assumes that the GDP in year n is given as the sum of three components,

$$y_n = c_n + g_n + i_n \,, \tag{90}$$

where c_n denotes the consumer spending, assumed to be proportional to the current economic climate as quantified by the GDP, g_n denotes the government spending, assumed constant, and i_n the induced investment, assumed to be proportional to the growth rate of the GDP during the previous year. Thus, we can write

$$y_n = \alpha\, y_n + \beta\,(y_{n-1} - y_{n-2}) + \gamma \,, \tag{91}$$

for some nonnegative parameters α, β, and γ with $\alpha < 1$.

The superposition principle for this model takes a very simple form. A particular solution is given by the equilibrium solution, where

$$y^* = \alpha\, y^* + \beta\,(y^* - y^*) + \gamma \,, \tag{92}$$

so that

$$y^* = \frac{\gamma}{1-\alpha} \,. \tag{93}$$

Two independent solutions to the homogeneous problem can be obtained via the method of generating functions, e.g. in the form of Theorem 6. We first illustrate this for the special case where $\alpha = \frac{1}{2}$ and $\beta = \gamma = 1$. Then $y^* = 2$ is the equilibrium solution, and the characteristic polynomial for the homogeneous equation is given by

$$p(x) = \tfrac{1}{2}\, x^2 - x + 1 \tag{94}$$

with roots

$$\lambda_{1,2} = 1 \pm \mathrm{i} = \sqrt{2}\, \mathrm{e}^{\mathrm{i}\pi/4} \,. \tag{95}$$

The general solution is therefore

$$y_n = 2 + c_1\, \lambda_1^n + c_2\, \lambda_2^n \,. \tag{96}$$

Since $\lambda_1 = \overline{\lambda}_2$, where $\overline{\lambda}_2$ denotes the complex conjugate of λ_2, the solution is real-valued if and only if $c_1 = \overline{c}_2$. Setting $c_1 = a + \mathrm{i}b$ and recalling that $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\,\sin\theta$, we then find that

$$y_n = 2 + 2\,\mathrm{Re}(c_1\, \lambda_1^n) = 2 + 2^{n/2+1}\,\bigl(a\,\cos(n\pi/4) - b\,\sin(n\pi/4)\bigr) \,, \tag{97}$$

where a and b are determined by the initial conditions. We see that the solution is oscillating about the equilibrium point with growing amplitude. Thus, the given parameter values describe an unstable economy.

More generally, the roots of the characteristic polynomial to the homogeneous part of (92) are given by

$$\lambda_{1,2} = \frac{\beta \pm \sqrt{\beta^2 - 4(1-\alpha)\beta}}{2(1-\alpha)} \,. \tag{98}$$

In the oscillatory case, when $4(1-\alpha) > \beta$, so that $\lambda_{1,2}$ are a pair of complex conjugate roots, we compute for $i = 1, 2$ that

$$|\lambda_i|^2 = \lambda_1 \lambda_2 = \frac{\beta}{1-\alpha}, \tag{99}$$

so that the amplitude of the oscillations is growing whenever $\beta > 1 - \alpha$, remains constant for $\beta = 1 - \alpha$, and decreases for $\beta < 1 - \alpha$. We conclude that the economy is stable if and only if $\beta \leq 1 - \alpha$.

The nonoscillatory case can be analyzed similarly; this shall be left as an exercise to the reader.

9 Equilibrium points for nonlinear equation of higher order

For nonlinear equation of higher order, explicit solutions are rare. Thus, we would like to extend the study of equilibrium points and cycles to this situation, so that at least some qualitative properties can be deduced in the absence of fully explicit solutions.

In analogy with the setting of Section 3, particularly equation (22), we consider constant coefficient nonlinear second order equations in the form

$$x_{n+2} = f(x_{n+1}, x_n). \tag{100}$$

An equilibrium point x^* is then characterized by

$$x^* = f(x^*, x^*). \tag{101}$$

We are again interested in the question of stability of such equilibrium points. To provide an intuitive argument, which could easily be made rigorous, we write

$$x_n = x^* + \delta_n \tag{102}$$

where we think of δ_n as an initially small perturbation to an equilibrium point; we want to study whether the perturbation δ_n grows or decreases. Plugging (102) into (100), we can obtain an approximate simpler expression on the right hand side by using a tangent plane approximation—the first two terms in a multivariate Taylor expansion—at the point x^*, whence

$$x^* + \delta_{n+2} \approx f(x^*, x^*) + \frac{\partial f(x^*, x^*)}{\partial x} \delta_{n+1} + \frac{\partial f(x^*, x^*)}{\partial y} \delta_n, \tag{103}$$

where $\partial f / \partial x$ denotes the derivative of f with respect to its first argument with the second argument held constant, and $\partial f / \partial y$ denotes the derivative of f with respect to its second argument with the first argument held constant. Due to (101), we obtain an approximate equation for δ_n, called the *linearization of* (100) *about the equilibrium point* x^*,

$$\delta_{n+2} = \frac{\partial f(x^*, x^*)}{\partial x} \delta_{n+1} + \frac{\partial f(x^*, x^*)}{\partial y} \delta_n. \tag{104}$$

This equation is a linear, second order, constant coefficient, homogeneous difference equation. Thus, the growth of the perturbation will depend on the absolute value of the roots of its characteristic polynomial. This motivates the following analog of Theorem 3.

Theorem 9. *Let x^* be an equilibrium point of the difference equation* (100) *with continuously differentiable right hand side f. Let λ_1 and λ_2 denote the roots of the characteristic polynomial of its linearization about the equilibrium point.*

(i) If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then x^* is an asymptotically stable equilibrium point.

(ii) If $|\lambda_1| > 1$ or $|\lambda_2| > 1$, then x^* is an unstable equilibrium point.

As for Theorem 3, there are cases where the criterion is inconclusive, namely when $|\lambda_i| \leq 1$ with identity for at least one of the roots.

A complete proof is more difficult, but essentially similar to the proof of Theorem 3.

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